Differential Geometry for Mesh Generation III

David Gu

Computer Science Department
Applied Mathematics Department
State University of New York at Stony Brook

gu@cs.stonybrook.edu

Short Course International Meshing Roundtable
SIAM IMR 2024, Baltimore, USA

March 5th, 2024

Structured Surface Quadrilateral Mesh Generation

Motivation

Spline Surfaces for IGA

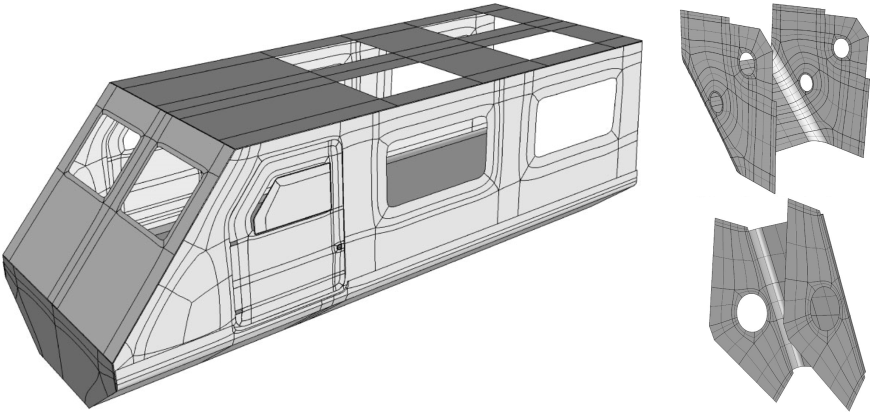


Figure: Bicubic spline representation of vehicle (joint work with Tom Hughes and K. Sheperd).

Spline Surfaces for IGA

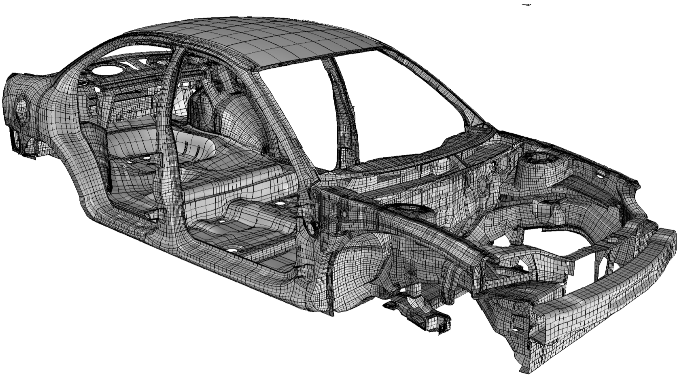
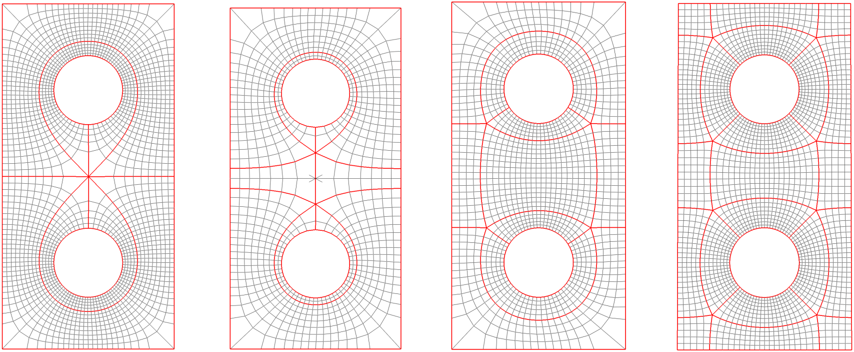


Figure: Dodge Neon model represented as bicubic set of NURBS splines (joint work with Tom Hughes and K. Sheperd).

Central Challenge

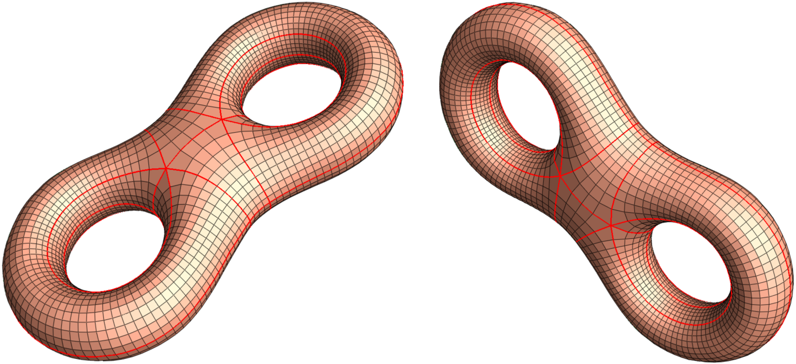


1 zero　　2 zeros　　4 zeros　　8 zeros

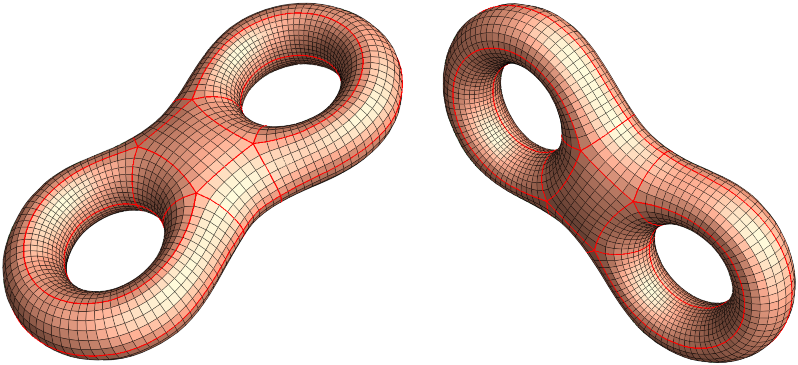
Problem (Central Task)

Find the governing equations for the singularities of a quad-mesh.

Central Challenge



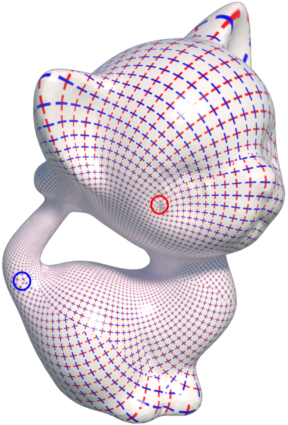
Central Challenge

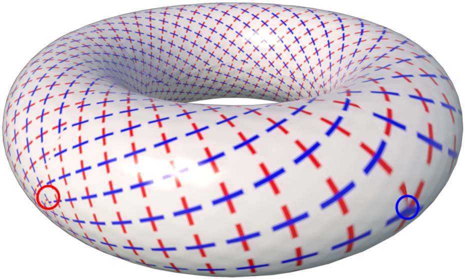


Problem (Central Task)

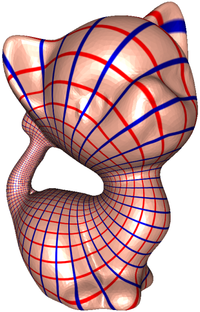
Find the governing equations for the singularities of a quad-mesh.

Singularities on a Topological Torus





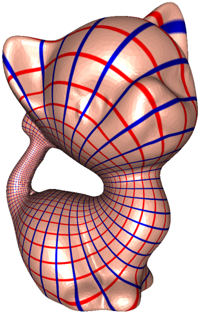
Smooth cross fields on genus one closed surfaces with two singularities.



Problem (3-5 Quad-Mesh on a Torus)

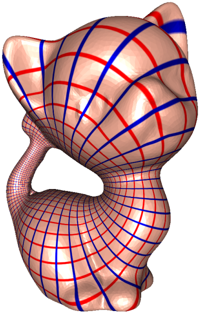
Is there a quad-mesh on a topological torus with one valence 3 singular point and one valence 5 singular point?

Singularities on a Topological Torus



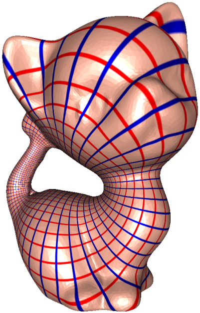
- There is no 3-5 quad-mesh;

Singularities on a Topological Torus

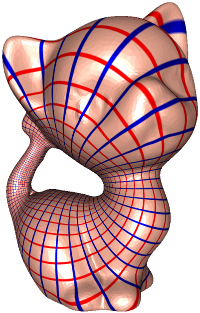


- There is no 3-5 quad-mesh;
- The combinatorial Euler formula is satisfied;

Singularities on a Topological Torus

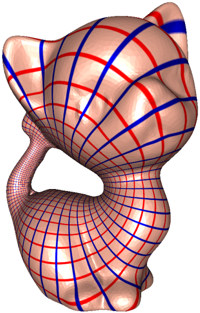


- There is no 3-5 quad-mesh;
- The combinatorial Euler formula is satisfied;
- There are flat metrics with 2 cone singularities, whose curvatures are $-\pi/2$ and $\pi/2$ respectively;



- There is no 3-5 quad-mesh;
- The combinatorial Euler formula is satisfied;
- There are flat metrics with 2 cone singularities, whose curvatures are $-\pi/2$ and $\pi/2$ respectively;
- There are cross fields with 2 singularities, whose indices are $-1/4$ and $1/4$ respectively;

Singularities on a Topological Torus



- There is no 3-5 quad-mesh;
- The combinatorial Euler formula is satisfied;
- There are flat metrics with 2 cone singularities, whose curvatures are $-\pi/2$ and $\pi/2$ respectively;
- There are cross fields with 2 singularities, whose indices are $-1/4$ and $1/4$ respectively;

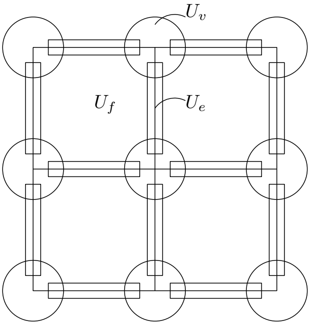
This means differential topolgy and Riemannian geometry are not enough for quad-mesh theory. Conformal geometry is essential.

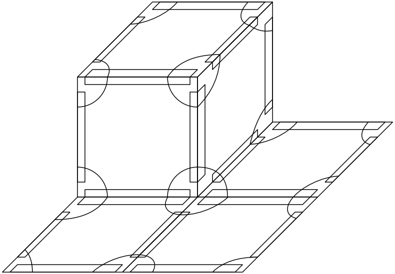
Quad-Mesh Metric Structure

Quad-Mesh Metric

Definition (Quad-Mesh Metric)

Given a quad-mesh $\mathcal{Q}$, each face is treated as the unit planar square, this will define a Riemannian metric, the so-called quad-mesh metric $\mathbf{g}_{\mathcal{Q}}$, which is a flat metric with cone singularities.





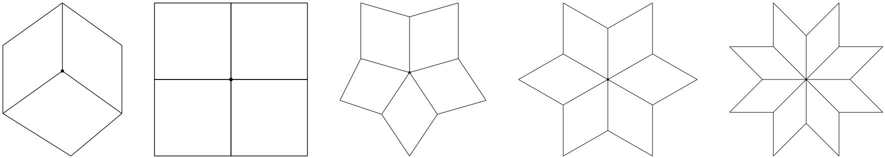
Discrete Gauss Curvature

Definition (Curvature)

Given a quad-mesh $\mathcal{Q}$, for each vertex v_i, the curvature is defined as

$$K(v) = \begin{cases} \frac{\pi}{2}(4 - k(v)) & v \notin \partial\mathcal{Q} \\ \frac{\pi}{2}(2 - k(v)) & v \in \partial\mathcal{Q} \end{cases}$$

where $k(v)$ is the topological valence of v, i.e. the number of faces adjacent to v.



$k = \pi/2$ $k = 0$ $k = -\pi/2$ $k = -\pi$ $k = -2\pi$

Quad-Mesh Metric Conditions

Theorem (Quad-Mesh Metric Conditions)

Given a quad-mesh $\mathcal{Q}$, the induced quad-mesh metric is $\mathbf{g}_{\mathcal{Q}}$, which satisfies the following four conditions:

1. *Gauss-Bonnet condition;*
2. *Holonomy condition;*
3. *Finite horizontal/vertical geodesic condition;*
4. *Boundary Alignment condition.*

1. Gauss-Bonnet Condition

Theorem (Gauss-Bonnet)

Given a quad-mesh $\mathcal{Q}$, the induced metric is $\mathbf{g}_{\mathcal{Q}}$, the total curvature satisfies

$$\sum_{v_i \in \partial \mathcal{Q}} K(v_i) + \sum_{v_i \notin \partial \mathcal{Q}} K(v_i) = 2\pi\chi(\mathcal{Q}).$$

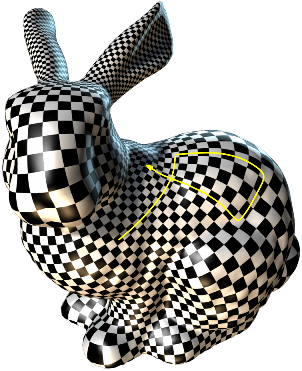
Namely

$$\sum_{v_i \in \partial \mathcal{Q}} (2 - k(v_i)) + \sum_{v_i \notin \partial \mathcal{Q}} (4 - k(v_i)) = 4\chi(\mathcal{Q}).$$

2. Holonomy Condition

Theorem (Holonomy Condition)

Suppose $\mathcal{Q}$ is a closed quad-mesh, then the holonomy group induced by $\mathbf{g}_{\mathcal{Q}}$ is a subgroup of the rotation group $\{e^{i\frac{k}{2}\pi}, k \in \mathbb{Z}\}$.



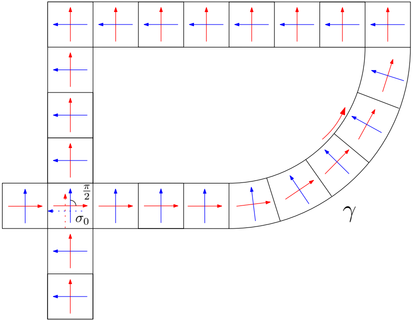


Figure: Parallel transportation along a face loop.

Definition (Boundary Alignment Condition)

Given a quad-mesh $\mathcal{Q}$, with induced metric $\mathbf{g}_{\mathcal{Q}}$, one can define a global cross field by parallel transportation, which is aligned with the boundaries.

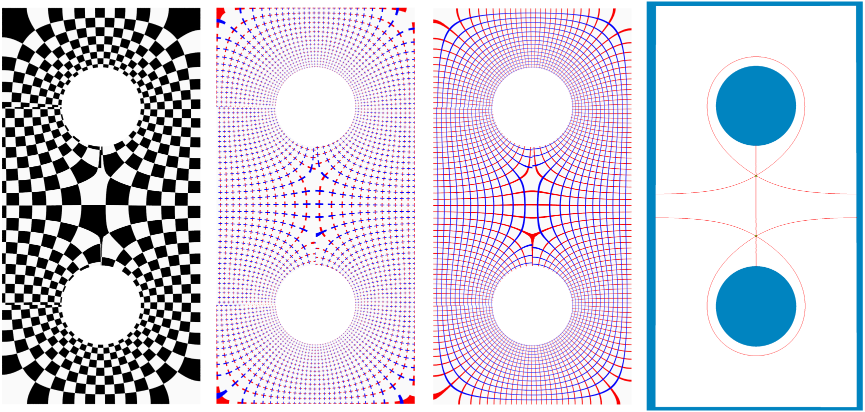


Figure: Aligned and mis-aligned with the inner boundaries.

3. Boundary Alignment Condition

Definition (Boundary Alignment Condition)

Given a quad-mesh $\mathcal{Q}$, with induced metric $\mathbf{g}_{\mathcal{Q}}$, one can define a global cross field by parallel transportation, which is aligned with the boundaries.

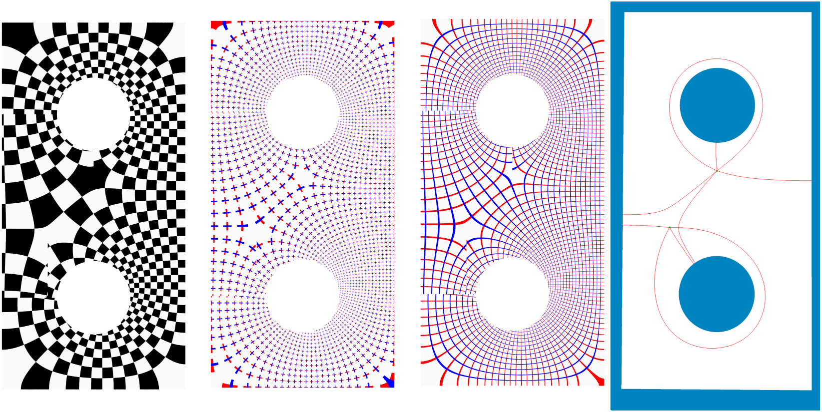
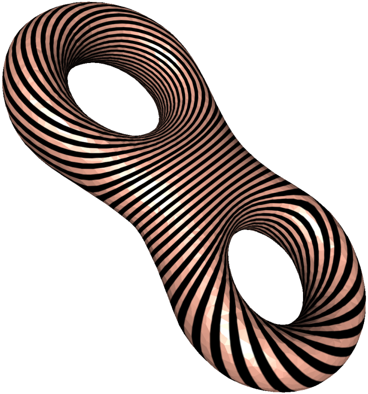


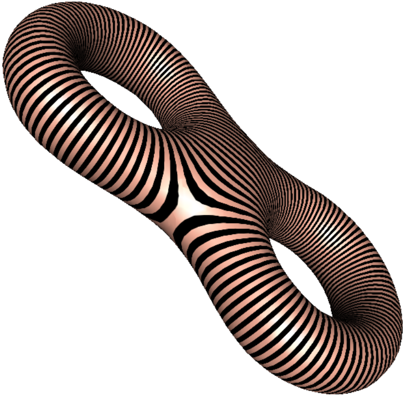
Figure: Aligned and mis-aligned with the inner boundaries.

4. Finite Horizontal/Vertical Geodesic Condition

Definition (Finite Horizontal/Vertical Geodesic Condition)

The stream lines parallel to the cross field are finite geodesic loops.





Riemann Surface Theory

Riemann Surface

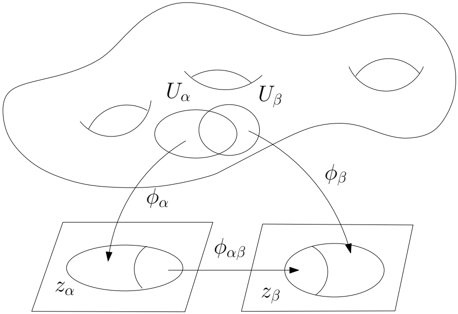


Figure: A Riemann surface.

A surface is covered by a complex atlas $\mathcal{A}$, such that all chart transitions are bi-holomorphic. $\varphi_{\alpha\beta} : (x, y) \mapsto (u, v)$ satisfies Cauchy-Riemann equation:

$$u_x = v_y, \quad u_y = -v_x,$$

Definition (Meromorphic Function)

Suppose $f : M \to \mathbb{C} \cup \{\infty\}$ is a complex function defined on the Riemann surface M. If for each point $p \in M$, there is a neighborhood $U(p)$ of p with local parameter $z(p) = 0$, f has Laurent expansion

$$f(z) = \sum_{i=k}^{\infty} a_i z^i,$$

then f is called a meromorphic function.

If all k's are non-negative, then f is a holomorphic function.

Meromorphic Differential

Definition (Meromorphic Differential)

Given a Riemann surface $(M, \{z_\alpha\})$, ω is a meromorphic differential of order n, if it has local representation,

$$\omega = f_\alpha(z_\alpha)(dz_\alpha)^n,$$

where $f_\alpha(z_\alpha)$ is a meromorphic function, n is an integer; if $f_\alpha(z_\alpha)$ is a holomorphic function, then ω is called a holomorphic differential of order n.

Zeros and Poles

Definition (Zeros and Poles)

Suppose $f : M \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function. For each point p, there is a neighborhood $U(p)$ of p with local parameter $z(p) = 0$, f has Laurent expansion

$$f(z) = \sum_{i=k}^{\infty} a_i z^i,$$

if $k > 0$, then p is a zero with order k; if $k = 0$, then p is a regular point; if $k < 0$, then p is a pole with order k. The assignment of p with respect to f is denoted as $\nu_p(f) = k$.

Divisor

Definition (Divisor)

The Abelian group freely generated by points on a Riemann surface is called the divisor group, every element is called a divisor, which has the form $D = \sum_p n_p p$. The degree of a divisor is defined as $deg(D) = \sum_p n_p$.

Definition (Meromorphic Function Divisor)

Given a meromorphic funciton f defined on a Riemann surface S, its divisor is defined as $(f) = \sum_p \nu_p(f)p$, where $\nu_p(f)$ is the assignment of p with respect to f.

The divisor of a meromorphic function is called a principle divisor.

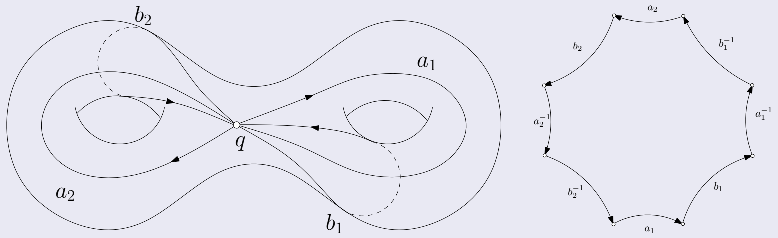
Principle Divisor

Theorem

Suppose M is a compact Riemann surface, f is a meromorphic function, then

$$\deg((f)) = 0.$$

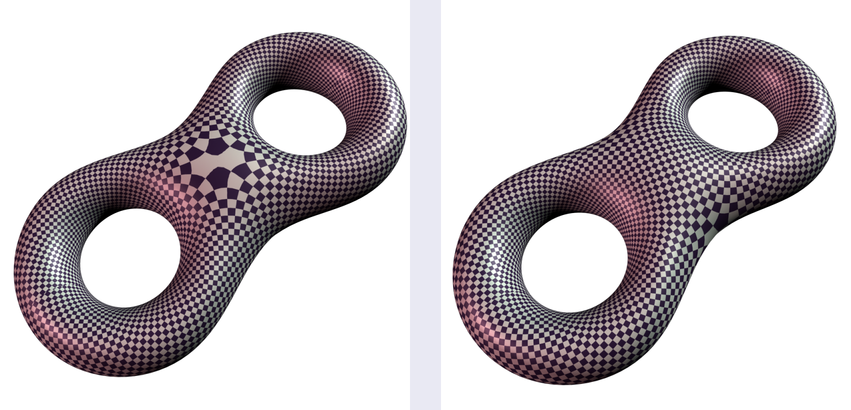
Canonical Fundamental Group Generators



Algebraic intersection numbers satisfy the conditions:

$$a_i \cdot b_j = \delta_{ij}, a_i \cdot a_j = 0, b_i \cdot b_j = 0.$$

Holomorphic Differential Group Basis



The holomorphic one-form basis $\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$ satisfy the dual condition

$$\int_{a_j} \varphi_i = \delta_{ij}.$$

Period Matrix

Definition (Period Matrix)

Suppose M is a compact Riemann surface of genus g, with canonical fundamental group basis

$$\{a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g\}$$

and holomorphic one form basis

$$\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$$

The period matrix is defined as $[A, B]$

$$A = \left(\int_{a_j} \varphi_i\right), B = \left(\int_{b_j} \varphi_i\right).$$

Jacobi Variety

Definition (Jacobi Variety)

Suppose the period matrix

$$A = (A_1, A_2, \cdots, A_g), \quad B = (B_1, B_2, \cdots, B_g),$$

the lattice Γ is

$$\Gamma = \left\{ \sum_{i=1}^{g} \alpha_i A_i + \sum_{j=1}^{g} \beta_j B_j \right\},$$

the Jacobi variety of M is defined as

$$J(M) = \mathbb{C}^g / \Gamma.$$

Jacobi Map

Definition (Jacobi Map)

Given a compact Riemann surface M, choose a set of canonical fundamental group generators $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$, and obtain a fundamental domain Ω,

$$\partial\Omega = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

choose a base point p_0, the Jacobi map $\mu : M \to J(M)$ is defined as follows: for any point $p \in M$, choose a path γ from p_0 to p inside Ω,

$$\mu(p) = \left(\int_\gamma \varphi_1, \int_\gamma \varphi_2, \cdots, \int_\gamma \varphi_g \right)^T.$$

Abel Theorem

Theorem (Abel)

Suppose M is a compact Riemann surface with genus g, D is a divisor, $\deg(D) = 0$. D is principle if and only if

$$\mu(D) = 0 \quad \text{in } J(M).$$

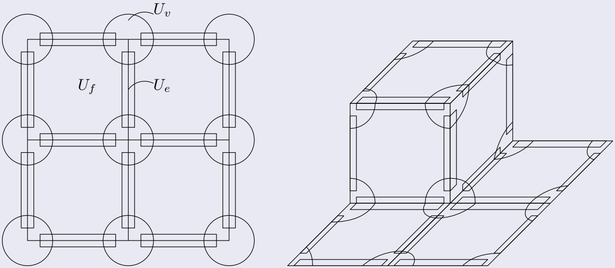
Quad-Mesh Conformal Structure

Quad-Mesh Riemann Surface

Theorem (Quad-Mesh Riemann Surface)

Suppose Q is a closed quadrilateral mesh, then Q induces a conformal structure and can be treated as a Riemann surface M_Q.

Proof.



(a) conformal atlas (b) singularities

$$z_e = z_f + \frac{1}{2}(\pm 1 \pm i), \qquad z_v^{\frac{k}{4}} = e^{i\frac{n\pi}{2}} z_f + \frac{1}{2}(\pm 1 \pm i) \tag{1}$$

Quad-Mesh Meromorphic Differential

Theorem (Quad-Mesh Meromorphic Differential)

Suppose Q is a closed quadrilateral mesh, then Q induces meromorphic quartic differential.

Proof.

On each face f, define dz_f, $\omega_Q = (dz_f)^4$; vertex face transition

$$z_v^{\frac{k}{4}} = e^{i\frac{n\pi}{2}} z_f + \frac{1}{2}(\pm 1 \pm i)$$

where k is the vertex valence, therefore

$$\left(\frac{k}{4}\right)^4 z_v^{k-4} (dz_v)^4 = (dz_f)^4 = \omega_Q. \quad (2)$$

□

Divisor

Definition (Divisor)

The Abelian group freely generated by points on a Riemann surface is called the divisor group, every element is called a divisor, which has the form $D = \sum_p n_p p$. The degree of a divisor is defined as $deg(D) = \sum_p n_p$. Suppose $D_1 = \sum_p n_p p$, $D_2 = \sum_p m_p p$, then $D_1 \pm D_2 = \sum_p (n_p \pm m_p) p$; $D_1 \leq D_2$ if and only if for all p, $n_p \leq m_p$.

Definition (Quad-Mesh Divisor)

Suppose Q is a closed quadrilateral mesh, then Q induces a divisor

$$D_Q = \sum_{v_i \in Q} (k(v_i) - 4) v_i,$$

where v_i is a vertex with valence $k(v_i)$.

Quad-Mesh Abel-Jacobi Condition

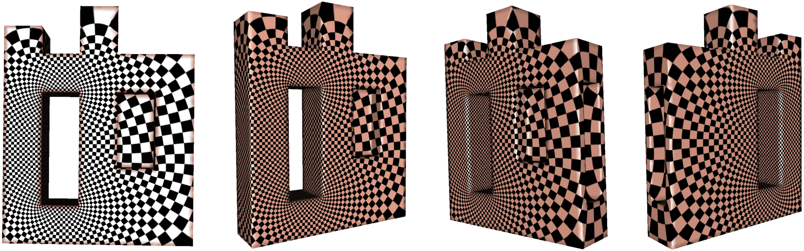
Theorem (Quad-Mesh Abel-Jacobi Condition 2020)

Suppose Q is a closed quadrilateral mesh, then for any holomorphic one-form φ

$$\mu(D_Q - 4(\varphi)) = 0 \quad \text{in } J(M_Q). \tag{3}$$

Genus One Polycube Surface Example

A genus one closed surface S, which is a polycube surface (union of canonical unit cubes). The holomorphic one form $\omega \in \Omega^1(S)$.

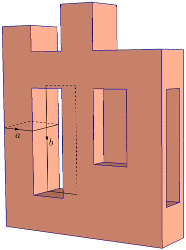


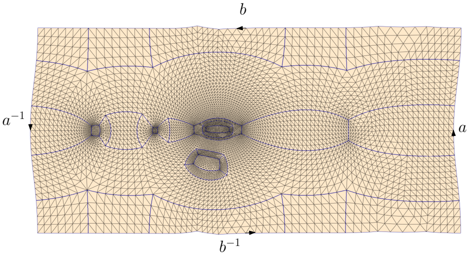
Genus One Polycube Surface Example

The homology basis is $\{a, b\}$, the surface is sliced along $\{a, b\}$ to get a fundamental domain D, $\partial D = abab^{-1}b^{-1}$. The conformal mapping $\mu : D \to \mathbb{C}$ is given by

$$\mu(q) = \int_p^q \omega,$$

where p is a base point and the integration path is arbitrarily chosen in D.

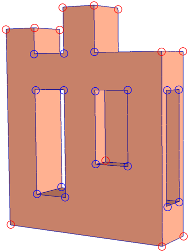


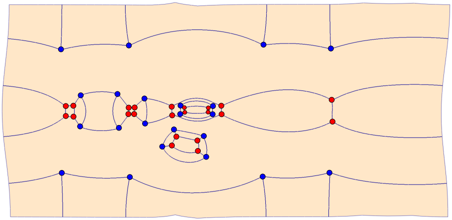


Genus One Polycube Surface Example

Suppose q_i's are poles (degree 3), p_j's are zeros (degree 5), then we have found that the number of poles equals to that of the zeros, furthermore,

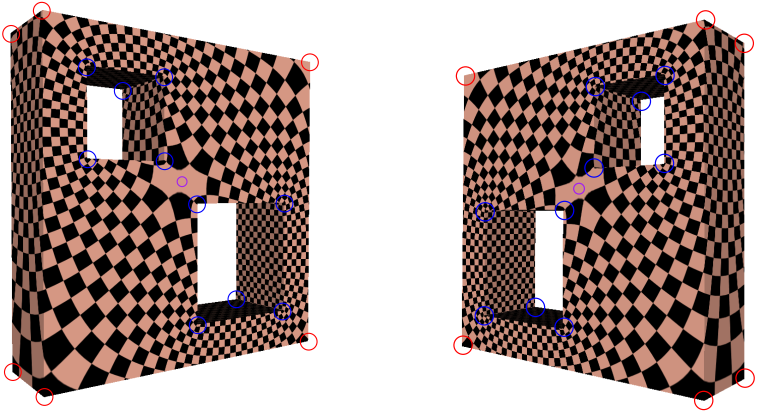
$$\sum_{j=1}^{22} \mu(p_j) - \sum_{i=1}^{22} \mu(q_i) = 0.$$





Genus Two Polycube Surface Example

Suppose S is a genus two polycube surface, ω is a holomorphic one-form. The red circles show the poles (degree 3), the blue circles show the zeros (degree 5), the purple circles the zeros of ω.



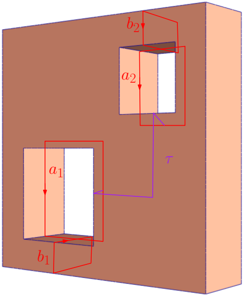
(a). front view (b). back view

Genus Two Polycube Surface Example

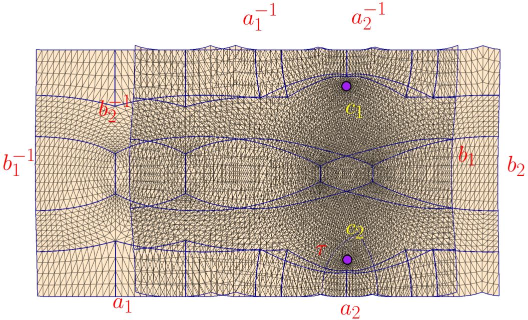
The surface is sliced along a_1, b_1, a_2, b_2, τ, and integrate ω to obtain $\mu : S \to \mathbb{C}$

$$\mu(q) = \int_p^q \omega,$$

it branch covers the plane, the branching points are zeros of ω, c_1, c_2.



(a). cuts

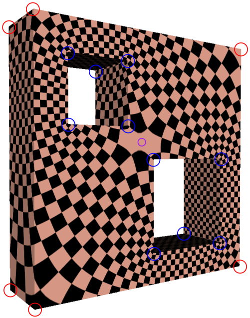


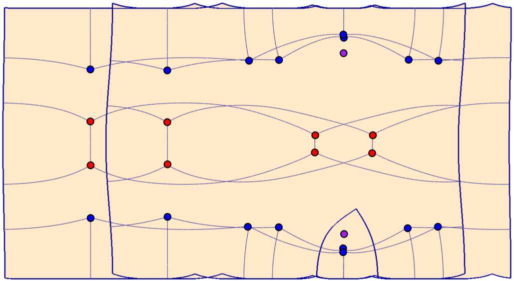
(b). conformal fattening

Genus Two Polycube Surface Example

Suppose p_i's are zeros (degree 5), q_j's are poles (degree 3), c_k's are branch points, then we have

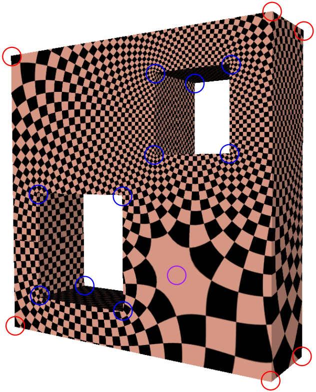
$$\sum_{i=1}^{16} \mu(p_i) - \sum_{j=1}^{8} \mu(q_j) = 4 \sum_{k=1}^{2} \mu(c_k).$$



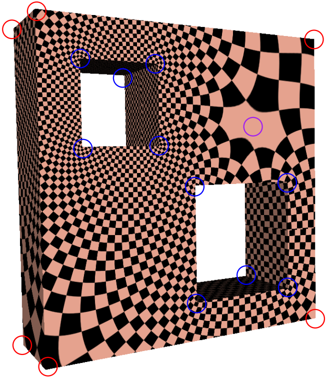


Genus Two Polycube Surface Example

Suppose S is a genus two polycube surface, ω is a holomorphic one-form. The red circles show the poles (degree 3), the blue circles show the zeros (degree 5), the purple circles the zeros of ω.



(a). front view

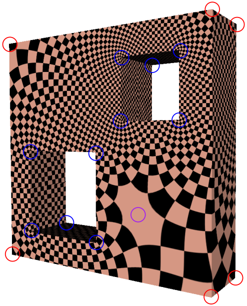


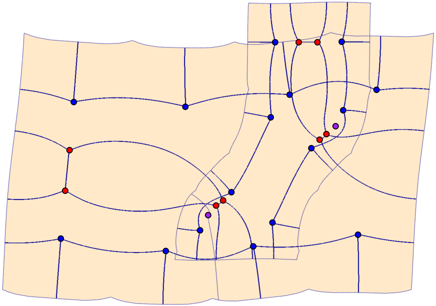
(b). back view

Genus Two Polycube Surface Example

Suppose p_i's are zeros (degree 5), q_j's are poles (degree 3), c_k's are branch points, then we have

$$\sum_{i=1}^{16}\mu(p_i) - \sum_{j=1}^{8}\mu(q_j) = 4\sum_{k=1}^{2}\mu(c_k).$$





Computational Algorithm

Algorithm Pipeline

Jacobi Map Algorithm

1. Compute the fundamental group $\pi_1(S, p)$ of the surface;
2. Compute the cohomology group basis $H_1(S, \mathbb{Z})$
3. Compute the harmonic form group basis $H_\Delta(S, \mathbb{R})$;
4. Compute the holomorphic 1-form group basis $\Omega^1(S)$;
5. Compute the Period Matrix of the surface.

Algorithm Pipeline

T-Mesh Generation Algorithm

1. Compute the singularity configuration by optimizing Abel-Jacobi condition;
2. Compute the flat cone metric using discrete surface Yamabe flow;
3. Compute the motorcycle graph;
4. Partition the surface into patches along the motorcycle graph, each patch is conformally flattened onto a quadrilateral;

Algorithm Pipeline

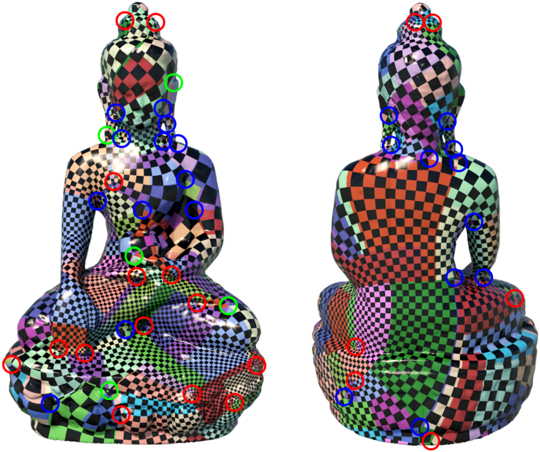


Figure: Step 1. Compute the singularities by optimizing Abel-Jacobi condition.

Algorithm Pipeline

Figure: Step 2. Compute the flat cone metric using surface Ricci flow, and compute the motorcycle graph.

Algorithm Pipeline

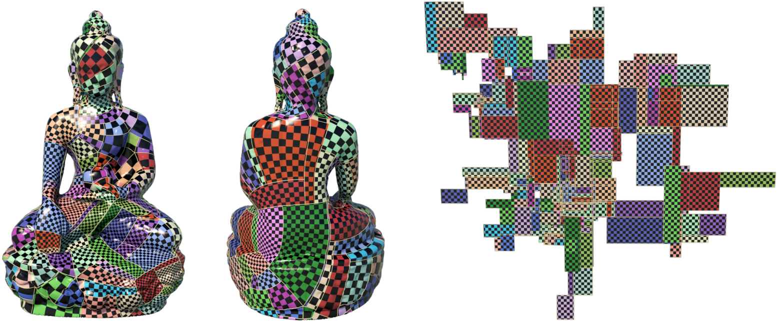


Figure: Step 3. Partition the surface into patches, each patch is conformally flattened onto a quadrilateral.

T-Meshes

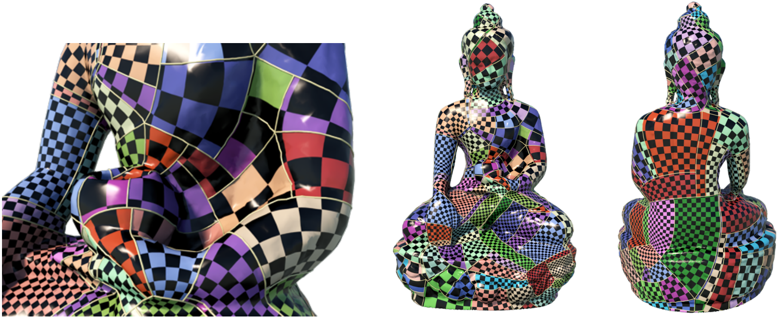


Figure: Step 4. Construct quad-meshes on each patch, with consistent boundary condition and adjust the width and the height of each quadrilateral.

T-Meshes

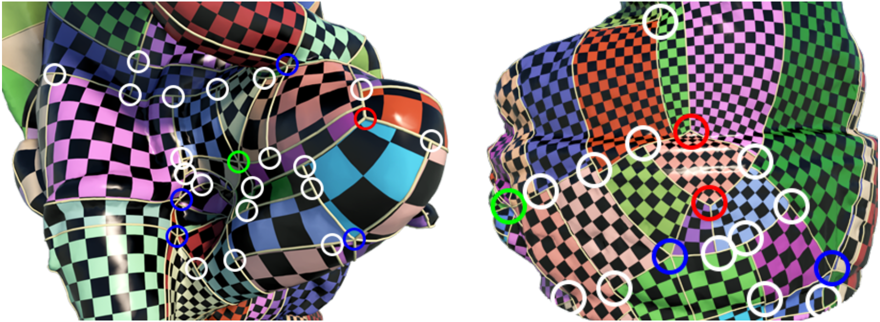


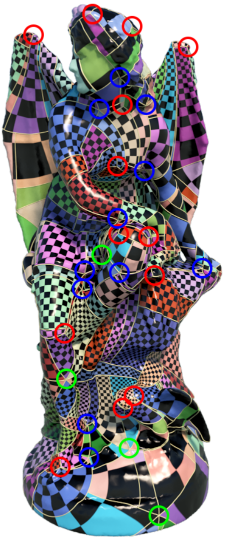
Figure: Singularities, white: T-junctions, blue: valence 5, green: valence 6, red :valence 3.

T-Meshes

Figure: Singularities and T-Mesh of the Loveme model.

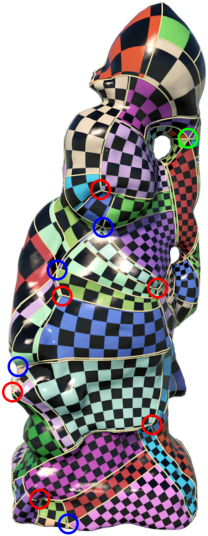
T-Meshes

T-Meshes



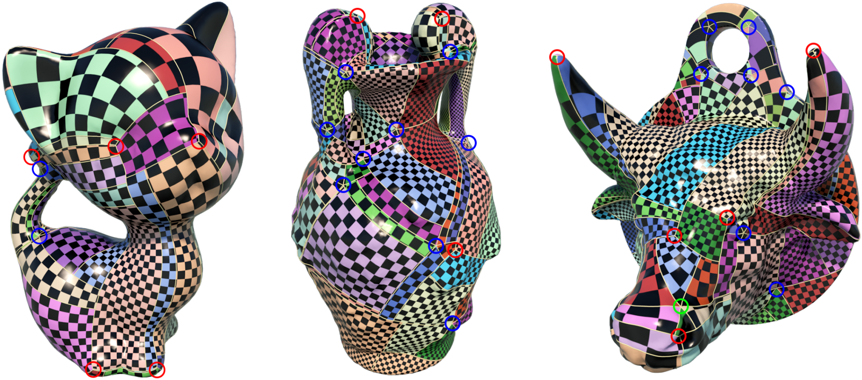
Witch model

Kiss model



Monk model

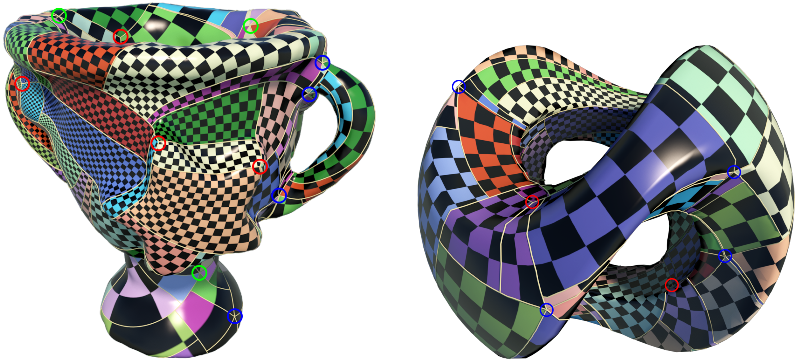
T-Meshes



(a) Kitten model (b) Amphora model (c) Bull head

Figure: Singularities and T-Meshes of various surfaces.

T-Meshes



Star cup model

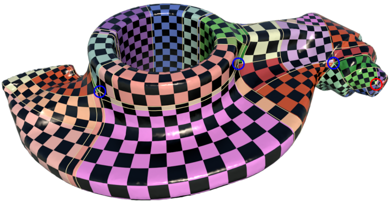
Sculpture model

Figure: Singularities and T-Meshes of high genus surfaces.

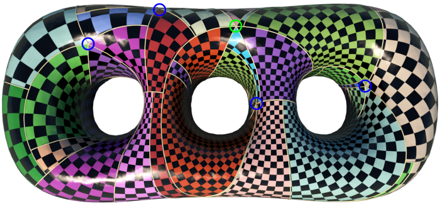
T-Meshes

Figure: The motorcycle graph and T-mesh of the genus 3 kiss model.

T-Meshes



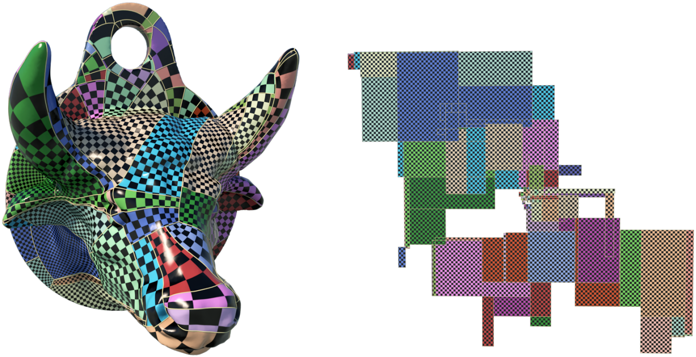
Rocker arm



3 holes surface

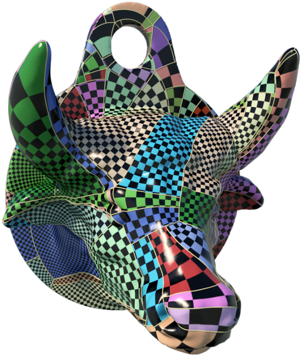
Figure: Singularities and T-Meshes of high genus surfaces.

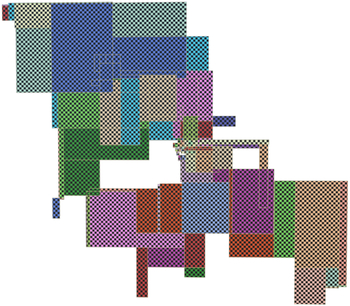
T-Mesh to Quad-Mesh



1. Puncture the surface at the singularities, isometrically immerse the universal covering space of the punctured surface obtain a fundamental polygon.

T-Mesh to Quad-Mesh





2. Deform the fundamental polygon, such that the translation components of all deck transformations are rational.

Quad-meshing Experimental Results

Quad-Meshes

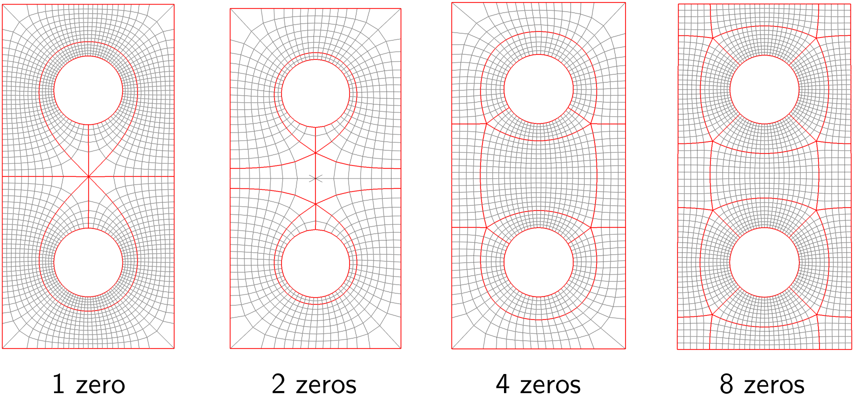


Figure: Quad-meshes of a planar domain with two holes.

Quad-Meshes

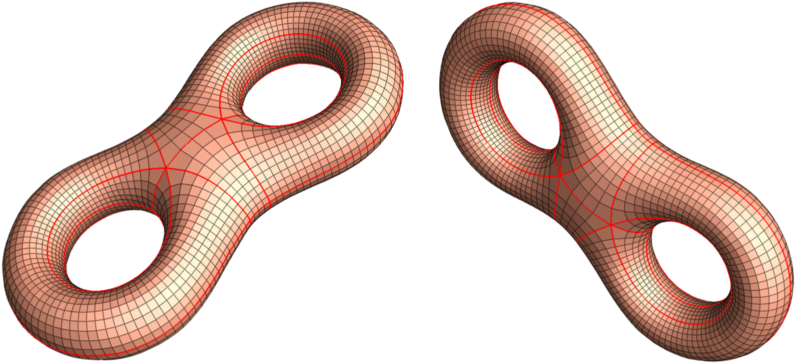


Figure: A quad-mesh of a genus two surface with 4 zeros.

Quad-Meshes

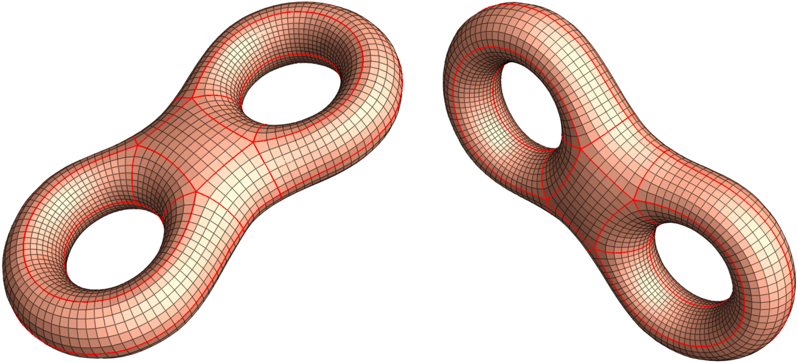
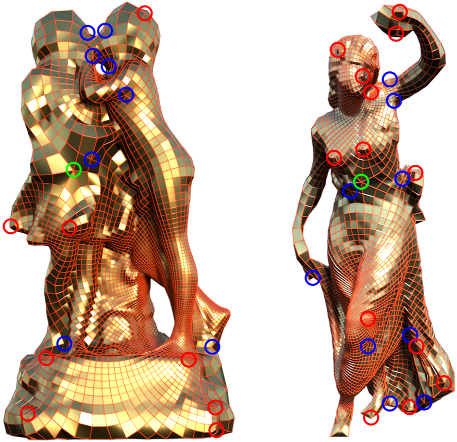


Figure: A quad-mesh of a genus two surface with 8 zeros.

Quad-Meshes



(a). genus 3　　(b). genus 1

Figure: Quad-Meshes.

Quad-Meshes

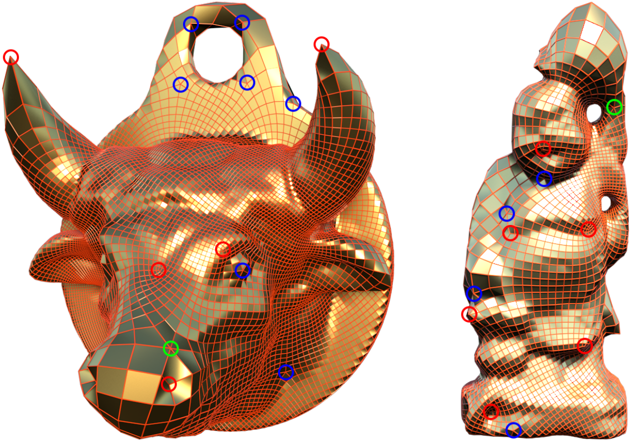


Figure: Quad-Meshes.

Quad-Meshes

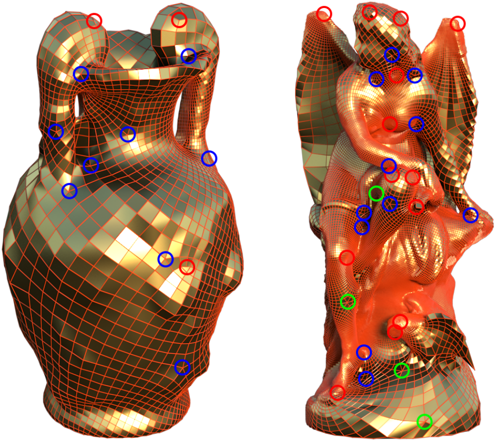
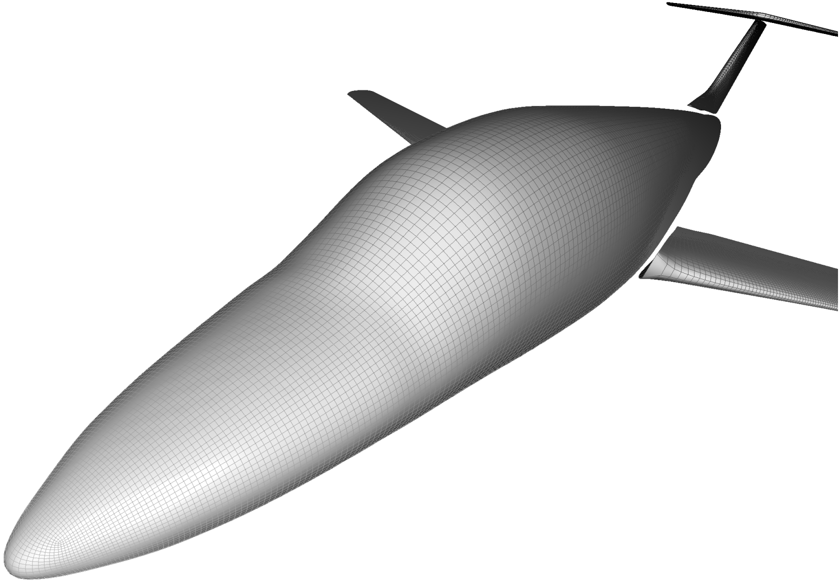
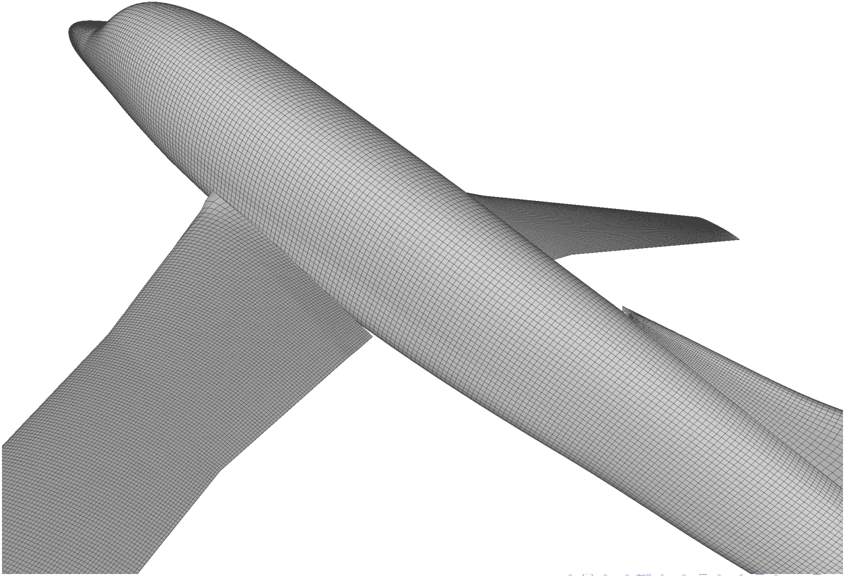


Figure: Quad-Meshes.

Quad-Meshes



Quad-Meshes



Spline Surfaces for IGA

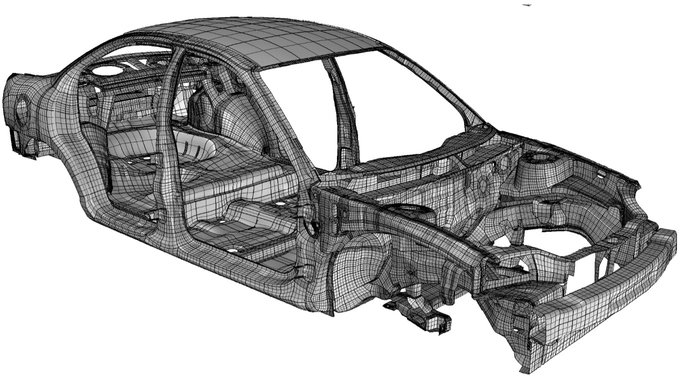


Figure: Dodge Neon model represented as bicubic set of NURBS splines (joint work with Tom Hughes and K. Sheperd).

IGA Application

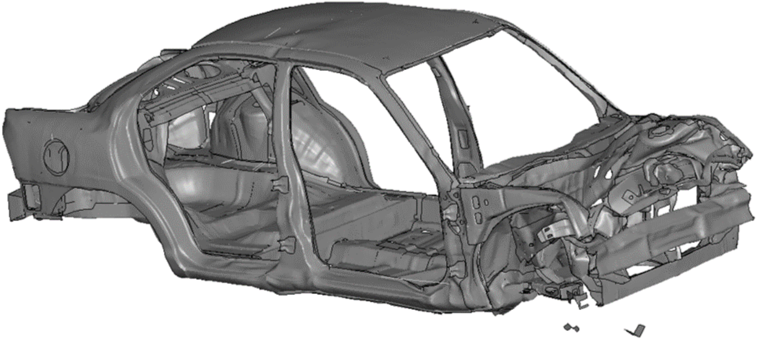


Figure: Crash analysis with Beta-CAE.

Automatically Generated Quad-Mesh

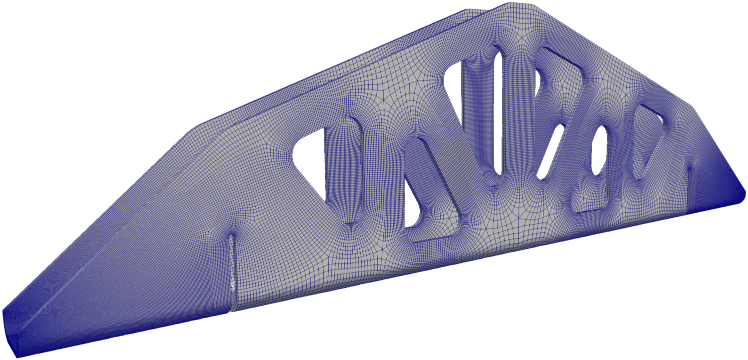


Figure: Devcom Stiffeners Bottom.

Automatically Generated Quad-Mesh

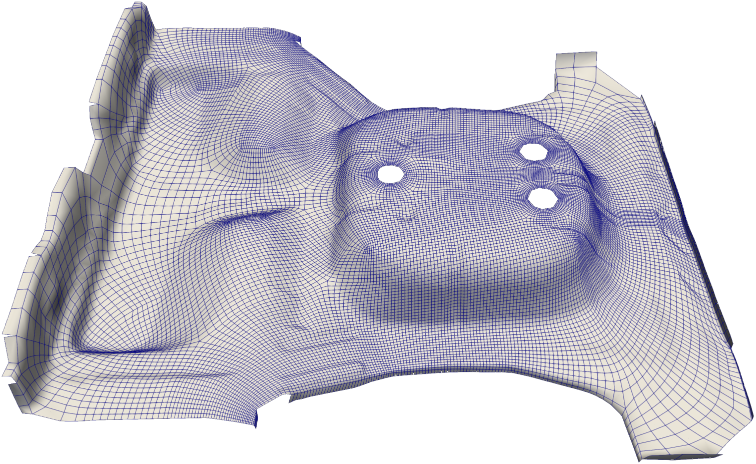


Figure: Floor board.

Automatically Generated Quad-Mesh

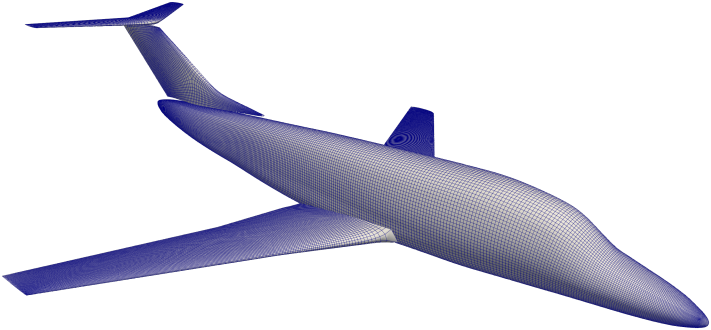


Figure: Air plane.

Automatically Generated Quad-Mesh

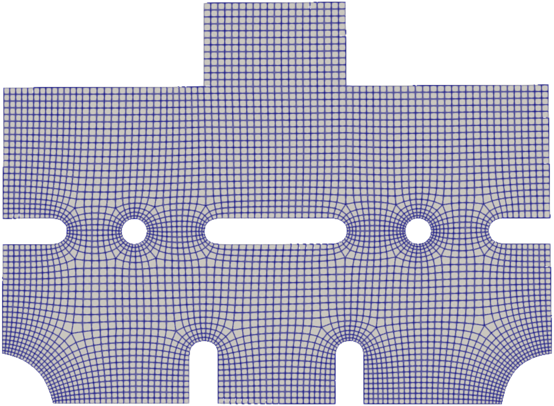


Figure: Industrial part.

References

- Xiaopeng Zheng, Yiming Zhu, Wei Chen, Na Lei, Zhongxuan Luo, Xianfeng Gu. **Quadrilateral Mesh Generation III : OptimizingSingularity Configuration Based on Abel-Jacobi Theory**. Computer Methods in Applied Mechanics and Engineering (CMAME), 2021.
- Na Lei, Xiaopeng Zheng, Zhongxuan Luo, Feng Luo and Xianfeng Gu, **Quadrilateral Mesh Generation II: Meoromorphic Quartic Differentials and Abel-Jacobi Condition**, Computer Methods in Applied Mechanics and Engineering (CMAME), 366(2020), 112980.
- Wei Chen, Xiaopeng Zheng, Jingyao Ke, Na Lei, Zhongxuan Luo; Xianfeng Gu, **Quadrilateral Mesh Generation I : Metric Based Method**, Computer Methods in Applied Mechanics and Engineering, Volume 356, Pages 652-668, 2019.

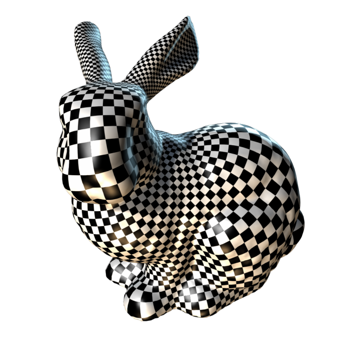
References

- Na Lei, Xiaopeng Zheng, Zhongxuan Luo, David Xianfeng Gu, **Quadrilateral and hexahedral mesh generation based on surface foliation theory II.**Computer Methods in Applied Mechanics and Engineering, Volume 321, Pages 406-426, July 2017.
- Na Lei, Xiaopeng Zheng, Jian Jiang, Yu-Yao Lin, David Xianfeng Gu, **Quadrilateral and hexahedral mesh generation based on surface foliation theory.** Computer Methods in Applied Mechanics and Engineering. Volume 316, Pages 758-781, April 2017.
- Emil Saucan and Xianfeng Gu. **Classical and Discrete Differential Geometry**, Publisher: CRC Press Taylor & Francis Group, POSTS & TELECOM Press 22 December 22, 2022. DOI: 10.1201/9781003350576. ISBN: 978-1-032-3907-8.

Conclusion

1. Bridge quadrilateral meshes and meromorphic quartic differentials; A global section of a holomorphic line bundle (4-th power of the cotangent bundel);
2. Singularities of a quad-mesh correspond to the divisor of the differential, which satisfies the Abel-Jacobi condition; characteristic class of the holomorphic line bundle;
3. T-mesh/Quad-mesh generation based on Abel-Jacobin condition and discrete surface Yamabe flow;

For more information, please email to gu@cs.stonybrook.edu.



Thank you!