Differential Geometry for Mesh Generation II

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Cross Field Construction

This part focuses on the construction of cross fields on surfaces, using Hodge decomposition and surface Ricci flow.

Simplicial Homology and Cohomology Group

Triangular mesh

Definition (triangular mesh)

A triangular mesh is a surface Σ with a triangulation T,

1. Each face is counter clockwisely oriented with respect to the normal of the surface.
2. Each edge has two opposite half-edges.

Simplicial Complex

Definition (Simplicial Complex)

Suppose $k+1$ points in the general positions in $\mathbb{R}^n$, $v_0, v_1, \cdots, v_k$, the standard simplex $[v_0, v_1, \cdots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \cdots, v_k] = \{x \in \mathbb{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\},$$

we call $v_0, v_1, \cdots, v_k$ as the vertices of the simplex σ.

Suppose $\tau \subset \sigma$ is also a simplex, then we say τ is a facet of σ.

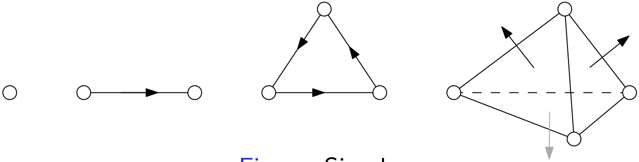


Figure: Simplex

Simplicial Complex

Definition (Simplicial complex)

A simplicial complex Σ is a union of simplices, such that

1. If a simplex σ belongs to Σ, then all its facets also belongs to Σ.
2. If $\sigma_1, \sigma_2 \subset \Sigma$, $\sigma_1 \cap \sigma_2 \neq \emptyset$, then their intersection is also a common facet.

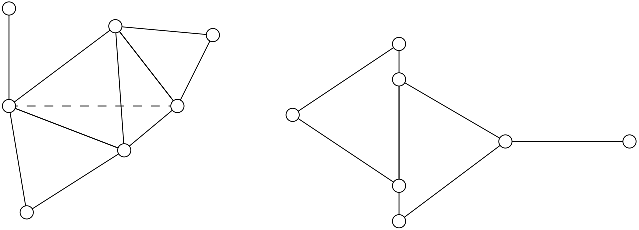


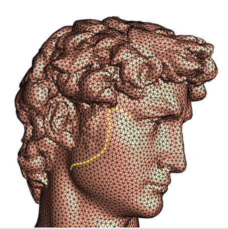
Figure: Simplicial complex.

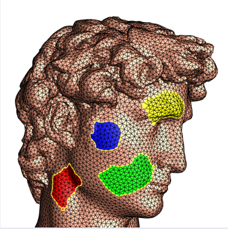
Chain Space

Definition (Chain Space)

A k chain is a linear combination of all k-simplicies in Σ, $\sigma = \sum_i \lambda_i \sigma_i, \lambda_i \in \mathbb{Z}$. The k dimensional chain space is the linear space formed by all k-chains, denoted as $C_k(\Sigma, \mathbb{Z})$.

A curve on the mesh is a 1-chain, a surface patch is a 2-chain.





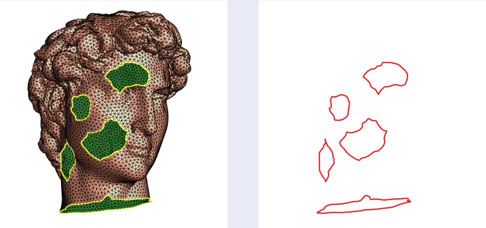
Boundary Operator

Definition (Boundary Operator)

The n-th dimensional boundary operator $\partial_n : C_n \to C_{n-1}$ is a linear operator, such that

$$\partial_n[v_0, v_1, v_2, \cdots, v_n] = \sum_i (-1)^i [v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n].$$

Boundary operator extracts the boundary of a chain.



Boundary Operator

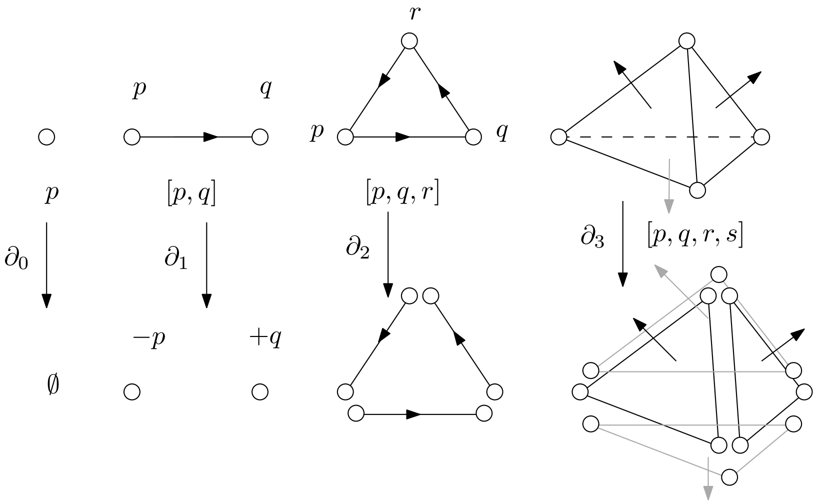


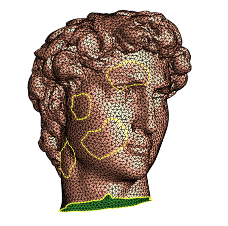
Figure: Boundary operator.

Closed Chains

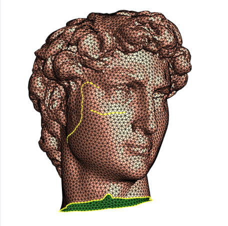
Definition (closed chain)

A k-chain $\gamma \in C_k(\sigma)$ is called a closed k-chain, if $\partial_k \gamma = 0$.

A closed 1-chain is a loop. A non-closed 1-chain has boundary vertices.



closed 1-chain

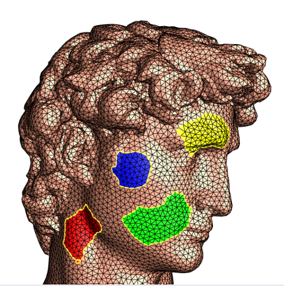


open 1-chain

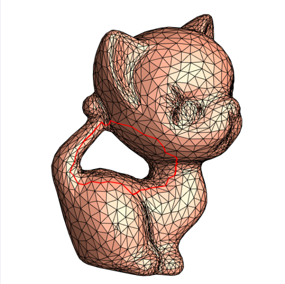
Exact Chains

Definition (Exact Chain)

A k-chain $\gamma \in C_k(\sigma)$ is called an exact k-chain, if there exists a $(k+1)$ chain σ, such that $\partial_{k+1}\sigma = \gamma$.



exact 1-chain



closed, non-exact 1-chain

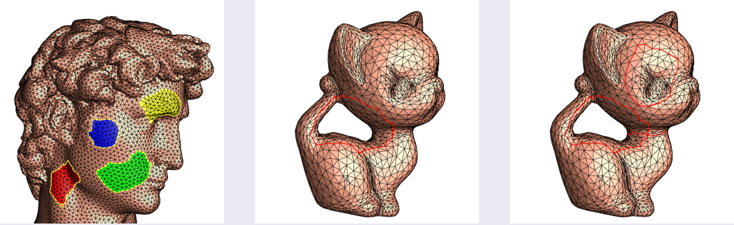
Boundary of Boundary

Theorem (Boundary of Boundary)

The boundary of a boundary is empty

$$\partial_k \circ \partial_{k+1} \equiv \emptyset.$$

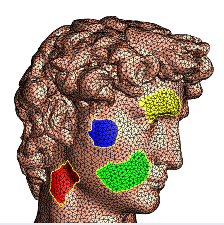
namely, exact chains are closed. But the reverse is not true.

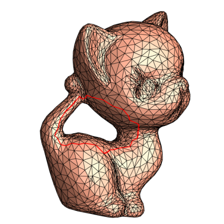


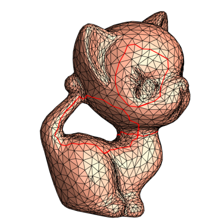
Homology

The difference between the closed chains and the exact chains indicates the topology of the surfaces.

1. Any closed 1-chain on genus zero surface is exact.
2. On tori, some closed 1-chains are not exact.







Homology Group

Closed k-chains form the kernel space of the boundary operator ∂_k. Exact k-chains form the image space of ∂_{k+1}.

Definition (Homology Group)

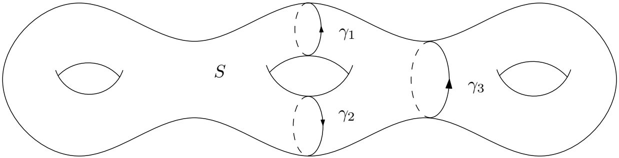
The k dimensional homology group $H_k(\Sigma, \mathbb{Z})$ is the quotient space of $ker\partial_k$ and the image space of $img\partial_{k+1}$.

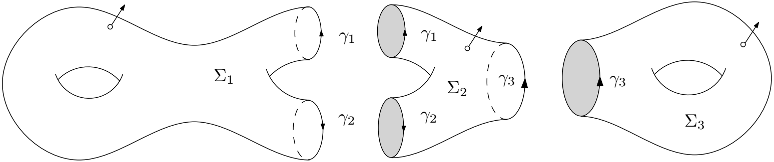
$$H_k(\Sigma, \mathbb{Z}) = \frac{ker\partial_k}{img\partial_{k+1}}.$$

Two k-chains γ_1,γ_2 are homologous, if they boundary a $(k+1)$-chain σ,

$$\gamma_1 - \gamma_2 = \partial_{k+1}\sigma.$$

Homological Classes





$$\partial\Sigma_1 = \gamma_1 - \gamma_2, \quad \partial\Sigma_2 = \gamma_3 - \gamma_1 + \gamma_2, \quad \partial\Sigma_3 = -\gamma_3.$$

γ_1 and γ_2 are not homotopic but homological; γ_3 is not homotopic to e, but homological to 0; γ_3 is homological to $\gamma_1 - \gamma_2$.

Homology vs. Homotopy

Abelianization

The first fundamental group in general is non-abelian. The first homology group is the abelianization of the fundamental group.

$$H_1(\Sigma) = \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)].$$

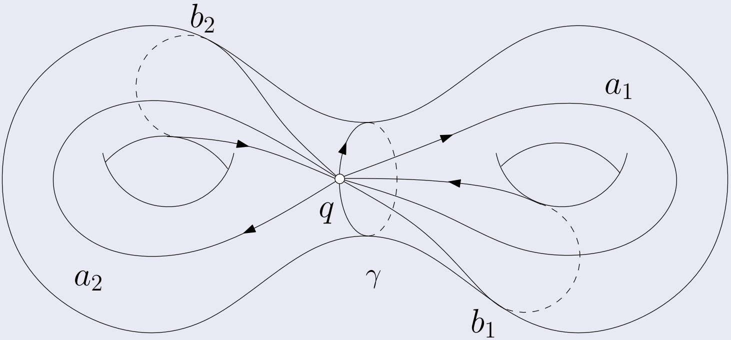
where $[\pi_1(\Sigma), \pi_1(\Sigma)]$ is the commutator of π_1,

$$[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}.$$

Fundamental group encodes more information than homology group, but more difficult to compute.

Homology vs. Homotopy

Homotopy group is non-abelian, which encodes more information than homology group.



- in homotopy group $\pi_1(S, q)$, $\gamma \sim [a, b]$,
- in homology group $H_1(S, \mathbb{Z})$, $\gamma \sim 0$.

Poincaré Duality

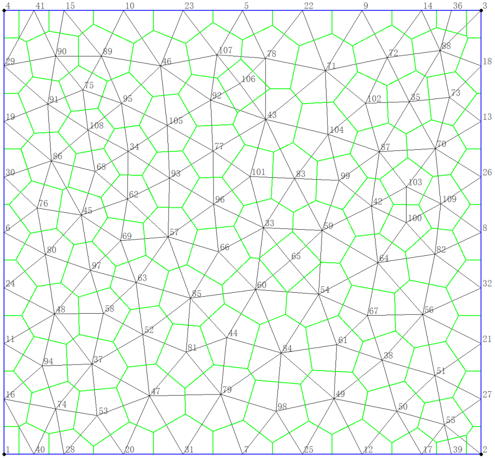
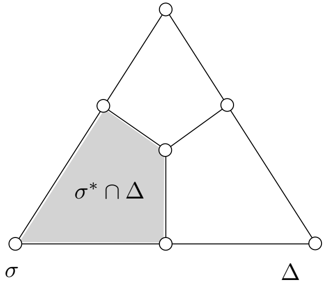


Figure: Poincaré Duality.

Poincaré Duality

Given a triangulated manifold T, there is a corresponding dual polyhedral decomposition T^*, which is a cell decomposition of the manifold such that the k-cells of T^* are in bijective correspondence with the $(n-k)$-cells of T.

Let σ be a simplex of T. Let Δ be a top-dimensional simplex of T containing σ, so we can think of σ as a subset of the vertices of Δ. Define the dual cell σ^* corresponding to σ so that $\Delta \cap \sigma^*$ is the convex hull in Δ of the barycentres of all subsets of the vertices of Δ that contain σ.



Homology Group

Theorem

Suppose M is a n dimensional closed manifold, then $H_k(M,\mathbb{Z}) \cong H_{n-k}(M,\mathbb{Z})$.

Proof.

The intersection map $C_k(T) \times C_{n-K}(T) \to \mathbb{Z}$ gives an isomorphism $C_k(T) \to C^{n-k}(T^*)$. □

Theorem

Suppose M is a genus g closed surface, then $H_0(M,\mathbb{Z}) \cong \mathbb{Z}$, $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^{2g}$, $H_2(M,\mathbb{Z}) \cong \mathbb{Z}$.

If $H_0(M,\mathbb{Z}) = \mathbb{Z}^k$, then M has k connected components.

Computation for Homology Basis

Each boundary operator: $\partial_k : C_k \to C_{k-1}$ is a linear map between linear spaces C_k and C_{k-1}, therefore it can be represented as a integer matrix. Suppose there are n_k k-simplexes of Σ, $\{\sigma_1^k, \sigma_2^k, \dots, \sigma_{n_k}^k\}$.

$$C_k = \left\{ \sum_{i=1}^{n_k} \lambda_i \sigma_i^k \right\}.$$

Boundary Matrix

The boundary matrix is defined as: $\partial_k = ([\sigma_i^{k-1}, \sigma_j^k])$, where

$$[\sigma_i^{k-1}, \sigma_j^k] = \begin{cases} +1 & +\sigma_i^{k-1} \in \partial_k \sigma_j^k \\ -1 & -\sigma_i^{k-1} \in \partial_k \sigma_j^k \\ 0 & \sigma_i^{k-1} \notin \partial_k \sigma_j^k \end{cases}$$

Computation for Homology Basis

Cominatorial Laplace Operator

Construct linear operator $\Delta_k : C_k \to C_k$,

$$\Delta_k := \partial_k^T \partial_k + \partial_{k+1}\partial_{k+1}^T,$$

the eigen vectors of zero eigen values of Δ_k form the basis of $H_k(M, \mathbb{Z})$.

Smith Norm

The eigen vectors can be found using Smith norm of integer matrix. The computational cost is very high.

Simplicial Cohomology Group

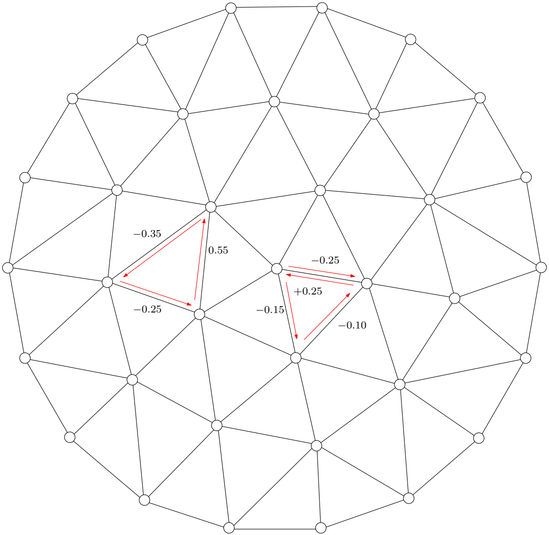


Figure: 1-Cochain.

Simplicial Cohomology Group

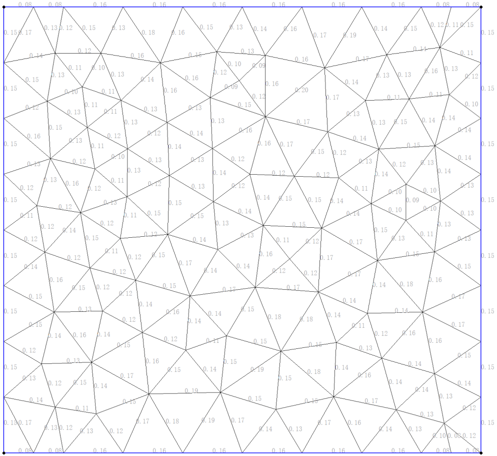


Figure: 1-Cochain.

Simplicial Cohomology Group

Definition (Cochain Space)

A k-cochain is a linear function

$$\omega : C_k \to \mathbb{Z}.$$

The k cochain space $C^k(\Sigma, \mathbb{Z})$ is a linear space formed by all the linear functionals defined on $C_k(\Sigma, \mathbb{Z})$. A k-cochain is also called a k-form.

Definition (Coboundary)

The coboundary operator $\delta_k : C^k(\Sigma, \mathbb{Z}) \to C^{k+1}(\Sigma, \mathbb{Z})$ is a linear operator, such that

$$\delta_k \omega := \omega \circ \partial_{k+1}, \omega \in C^k(\Sigma, \mathbb{Z}).$$

Simplicial Cohomology Group

Example

M is a 2 dimensional simplicial complex, ω is a 1-form, then $\delta_1\omega$ is a 2-form, such that

$$\begin{aligned}\delta_1\omega([v_0, v_1, v_2]) &= \omega(\partial_2[v_0, v_1, v_2]) \\ &= \omega([v_0, v_1]) + \omega([v_1, v_2]) + \omega([v_2, v_0])\end{aligned}$$

Cohomology

Coboundary operator is similar to differential operator. δ_0 is the gradient operator, δ_1 is the curl operator.

Definition (closed forms)

A k-form is closed, if $\delta_k \omega = 0$.

Definition (Exact forms)

A k-form is exact, if there exists a $k-1$ form σ, such that

$$\omega = \delta_{k-1}\sigma$$

Cohomology

suppose $\omega \in C^k(\Sigma)$, $\sigma \in C_k(\Sigma)$, we denote the pair

$$\langle \omega, \sigma \rangle := \omega(\sigma).$$

Theorem (Stokes)

$$\langle d\omega, \sigma \rangle = \langle \omega, \partial\sigma \rangle.$$

Theorem

$$\delta^k \circ \delta^{k-1} \equiv 0.$$

All exact forms are closed. The curl of gradient is zero.

Cohomology

The difference between exact forms and closed forms indicates the topology of the manifold.

Definition (Cohomology Group)

The k-dimensional cohomology group of Σ is defined as

$$H^n(\Sigma, \mathbb{Z}) = \frac{ker\delta^n}{img\delta^{n-1}}.$$

Two 1-forms ω_1, ω_2 are cohomologous, if they differ by a gradient of a 0-form f,

$$\omega_1 - \omega_2 = \delta_0 f.$$

Homology vs. Cohomology

Duality

$H_1(\Sigma)$ and $H^1(\Sigma)$ are dual to each other. suppose ω is a closed 1-form, σ is a closed 1-chain, then the pair $\langle \omega, \sigma \rangle$ is a bilinear operator.

Definition (dual cohomology basis)

suppose a homology basis of $H_1(\Sigma)$ is $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, the dual cohomology basis is $\{\omega_1, \omega_2, \cdots, \omega_n\}$, if and only if

$$\langle \omega_i, \gamma_j \rangle = \delta_i^j.$$

Cohomology was introduced by H. Whitney in order to represent stiefel whitney class characteristic class. Prof. Chern learned it from Whitney.

Algorithm for Cohomology Group

Algorithm for $H^1(M, \mathbb{R})$

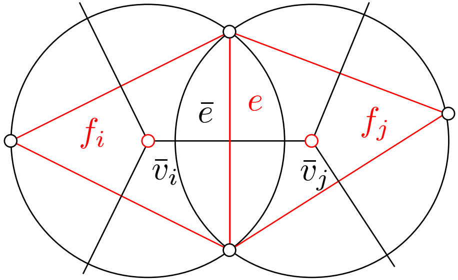
Input: A genus g closed triangle mesh M;

Output: A set of basis of $H^1(M, \mathbb{R})$

1. Compute a set of basis of $H_1(M, \mathbb{Z})$, denoted as
$$\{\gamma_1, \gamma_2, \cdots, \gamma_{2g}\},$$
2. for each γ_i, slice M along γ_i, to obtain a mesh with two boundaries M_i, $\partial M_i = \gamma_i^+ - \gamma_i^-$;
3. set a 0-form τ_i on M_i, such that $\tau_i(v) = 1$ for all $v \in \gamma_i^+$ and $\tau_i(w) = 0$, for all $w \in \gamma_i^-$; set $\omega_i = d\tau_i$;
4. All $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$ form a basis of $H^1(M, \mathbb{R})$.

Hodge Decomposition

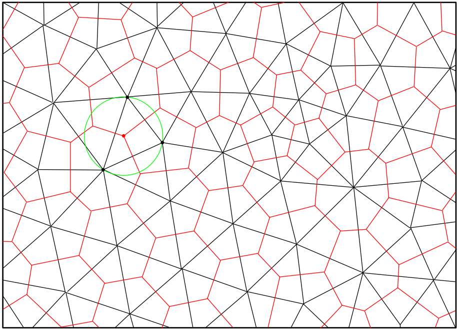
Discrete Hodge Operator



Cotangent edge weight:

$$w_{ij} = \frac{1}{2}(\cot\alpha + \cot\beta) \quad (1)$$

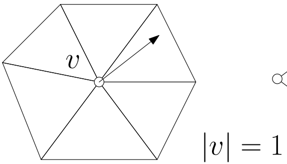
Dual Mesh



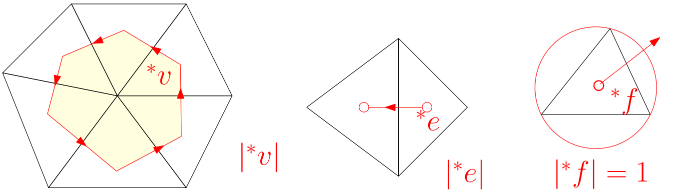
Poincaré's duality, equivalent to Delaunay triangulation and Voronoi diagram. The Delaunay triangulation is the primal mesh, the Voronoi diagram is the dual mesh.

Duality

$0 - form\ \eta$ $\quad$ $1 - form\ \omega$ $\quad$ $2 - form\ \Omega$



$$\frac{\eta(v)}{|v|} = \frac{{}^*\eta({}^*v)}{|{}^*v|} \qquad \frac{\omega(e)}{|e|} = \frac{{}^*\omega({}^*e)}{|{}^*e|} \qquad \frac{\Omega(f)}{|f|} = \frac{{}^*\Omega({}^*f)}{|{}^*f|}$$



Discrete Operator

Discrte Codifferential Operator

The codifferential operator $\delta : \Omega^p \to \Omega^{p-1}$ on an n-dimensional manifold,

$$\delta := (-1)^{n(p+1)+1} \, {}^*d^*.$$

Discrte Hodge star operator

$^{**} : \Omega^p \to \Omega^p$,

$$^{**} := (-1)^{(n-p)p}$$

$$\frac{\omega(e)}{|e|} = \frac{{}^*\omega({}^*e)}{|{}^*e|} = \frac{{}^{**}\omega({}^{**}e)}{|{}^{**}e|} = \frac{{}^{**}\omega(-e)}{|-e|} = -\frac{{}^{**}\omega(e)}{|e|}$$

Therefore ${}^{**}\omega(e) = -\omega(e)$, this verifies when $n = 2, p = 1$, $^{**} = -1$.

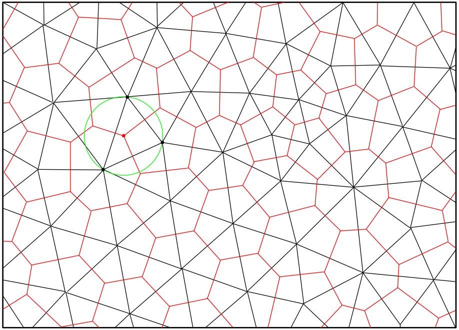
Harmonic 1-form

Definition (Harmonic 1-form)

Suppose ω is a 1-form, ω is harmonic iff

$$d\omega = 0, \quad \delta\omega = 0.$$

Dual Mesh



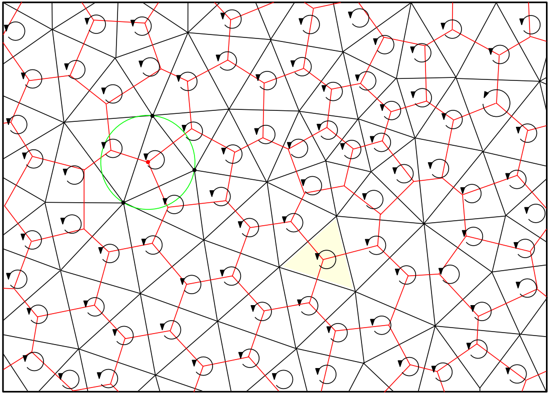
Theorem (Hodge Decomposition)

Suppose ω is a one-form on the prime mesh, it has the unique decomposition:

$$\omega = d\eta + \delta\Omega + h$$

where η is a 0-form, Ω a 2-form and h a harmonic one-form.

Discrete Harmonic One-form



compute $d\omega$,

$$d\omega = d^2\eta + d\delta\Omega + dh = d\delta\Omega, \quad \Omega = (d\delta)^{-1}(d\omega).$$

δ^2 operator

Lemma

The operator $\delta^2 : \Omega^2 \to \Omega^1$ on a surface, has the following formula:

$$\delta^2\Omega([v_i, v_j]) = \frac{1}{w_{ij}}\left(\frac{\Omega(f_\Delta)}{|f_\Delta|} - \frac{\Omega(f_k)}{|f_k|}\right) \tag{2}$$

Proof.

$$\begin{aligned} \delta^2 &= (-1)^{n(p+1)+1} {}^*d^* = (-1)^1({}^*d^{0*}) \\ \delta^2\Omega([v_i, v_j]) &= (-1)({}^*d^{0*})\Omega([v_i, v_j]) \\ \frac{({}^*d^{0*})\Omega([v_i, v_j])}{|[v_i, v_j]|} &= \frac{({}^{**}d^{0*})\Omega({}^*[v_i, v_j])}{|{}^*[v_i, v_j]|} = -\frac{(d^{0*})\Omega([{}^*f_k, {}^*f_\Delta])}{|[{}^*f_k, {}^*f_\Delta]|} \end{aligned} \tag{3}$$

□

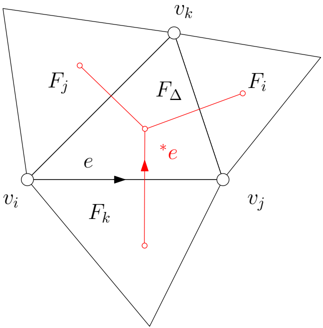
Proof.

$$
\begin{aligned}
&-\frac{(d^{0*})\Omega([{}^*f_k, {}^*f_\Delta])}{|[{}^*f_k, {}^*f_\Delta]|} = -\frac{{}^*\Omega(\partial_1[{}^*f_k, {}^*f_\Delta])}{|[{}^*f_k, {}^*f_\Delta]|} = -\frac{{}^*\Omega({}^*f_\Delta - {}^*f_k)}{|[{}^*f_k, {}^*f_\Delta]|} \\
&= -\frac{{}^*\Omega({}^*f_\Delta) - {}^*\Omega({}^*f_k)}{|[{}^*f_k, {}^*f_\Delta]|}
\end{aligned} \tag{4}
$$

$$
\frac{\Omega(f_\Delta)}{|f_\Delta|} = \frac{{}^*\Omega({}^*f_\Delta)}{|{}^*f_\Delta|} = {}^*\Omega({}^*f_\Delta), \frac{\Omega(f_k)}{|f_k|} = \frac{{}^*\Omega({}^*f_k)}{|{}^*f_k|} = {}^*\Omega({}^*f_k), \tag{5}
$$

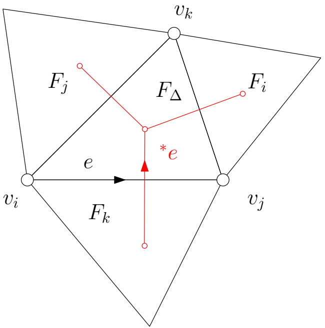
Plug (5) into (4), then plug (4) to (3), obtain the formula (2). □

Discrete Harmonic One-form



$$
\begin{aligned}
&\delta\Omega([v_i, v_j]) \\
=&(-1)(^*d^*)\Omega([v_i, v_j]) \\
=&(-1)(d^*\Omega)(^*[v_i, v_j])\frac{1}{w_{ij}}(-1) \\
=&\frac{1}{w_{ij}}(d^*\Omega)([^*f_k, {^*f_\Delta}]) \\
=&\frac{1}{w_{ij}}(^*\Omega)(\partial[^*f_k, {^*f_\Delta}]) \\
=&\frac{1}{w_{ij}}\left\{^*\Omega(^*f_\Delta) - {^*\Omega}(^*f_k)\right\} \\
=&\frac{1}{w_{ij}}\left\{\frac{\Omega(f_\Delta)}{|f_\Delta|} - \frac{\Omega(f_k)}{|f_k|}\right\}
\end{aligned}
$$

Discrete Harmonic One-form



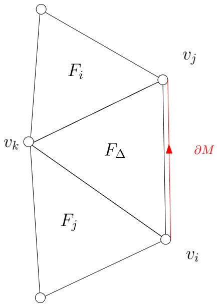
$$\delta\Omega([v_i, v_j]) = \frac{1}{w_{ij}} \left\{ \frac{\Omega(f_\Delta)}{|f_\Delta|} - \frac{\Omega(f_k)}{|f_k|} \right\}$$

For each face Δ, we have the equation $d\omega(\Delta) = \omega(\partial\Delta) = d\delta\Omega(\Delta)$,

$$\omega(\partial\Delta) = \frac{F_i - F_\Delta}{w_{jk}} + \frac{F_j - F_\Delta}{w_{ki}} + \frac{F_k - F_\Delta}{w_{ij}} \qquad (6)$$

where $F_i = -\frac{\Omega(f_i)}{|f_i|}$'s are 2-forms, ω is 1-form, w_{ij}'s are cotangent edge weights.

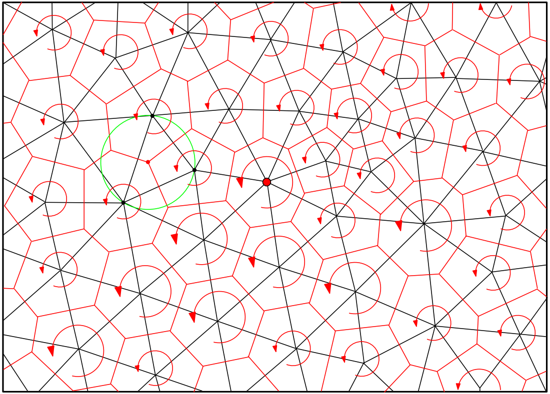
Discrete Harmonic One-form



For each boundary face Δ, we have the equation

$$d\omega(\Delta) = \omega(\partial\Delta) = \frac{F_i - F_\Delta}{w_{jk}} + \frac{F_j - F_\Delta}{w_{ki}} + \boxed{\frac{0 - F_\Delta}{w_{ij}}} \tag{7}$$

Discrete Harmonic One-form



compute $\delta\omega$,

$$\delta\omega = \delta d\eta + \delta^2\Omega + \delta h = \delta d\eta, \quad \eta = (\delta d)^{-1}(\delta\omega).$$

Discrete Harmonic One-form

Lemma

Suppose $\delta^1 : \Omega^1 \to \Omega^0$ *on a surface, then*

$$\delta^1\omega(v_i) = (-1)\frac{1}{|^* v_i|}\sum_j w_{ij}\omega([v_i, v_j]).$$

Proof.

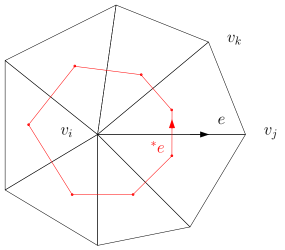
$$\delta^1 = (-1)^{n(p+1)+1} {}^* d^* = (-1)^{2(1+1)+1} {}^* d^{1*} = (-1)^* d^{1*},$$

$$\delta^1\omega(v_i) = (-1)(*d*)\omega(v_i) = (-1)*\underline{(d^1*)\omega}((v_i)_0) = (-1)\frac{1}{|*v_i|}\underline{(d^1*)\omega}((*v_i)_2)$$

$$= (-1)\frac{1}{|*v_i|}d^1\underline{(*\omega)}(*v_i) = (-1)\frac{1}{|*v_i|}\underline{(*\omega)}(\partial_2({}^*v_i))$$

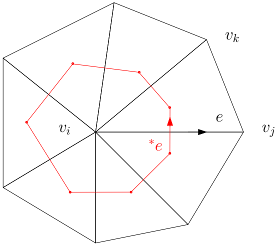
$$= (-1)\frac{1}{|*v_i|}\sum_j\underline{(*\omega)}(*[v_i, v_j]) = (-1)\frac{1}{|*v_i|}\sum_j(*\underline{\omega})(*[v_i, v_j])$$

$$= (-1)\frac{1}{|*v_i|}\sum_j w_{ij}\omega([v_i, v_j])$$



$$
\begin{aligned}
& \delta\omega(v_i) \\
&= (-1)(^*d^*)\omega(v_i) \\
&= (-1)(d^*\omega)(^*v_i)\frac{1}{|^*v_i|} \\
&= (-1)(^*\omega)(\partial\ ^*v_i)\frac{1}{|^*v_i|} \\
&= (-1)\sum_j (^*\omega)(^*e_{ij})\frac{1}{|^*v_i|} \\
&= (-1)\frac{1}{|^*v_i|}\sum_j w_{ij}\ \omega(e_{ij})
\end{aligned}
$$

Discrete Harmonic One-form



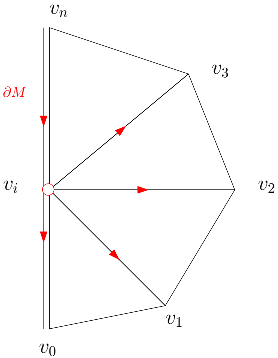
$$\delta\omega(v_i) = (-1)\frac{1}{|{}^*v_i|}\sum_j w_{ij}\;\omega(e_{ij})$$

For each vertex v_i, we obtain an equation $\delta\omega(v_i) = \delta d\eta(v_i)$,

$$\sum_{v_i \sim v_j} w_{ij}\;\omega([v_i, v_j]) = \sum_{v_i \sim v_j} w_{ij}(\eta_j - \eta_i). \tag{8}$$

where η_i's are 0-forms, w_{ij}'s are cotangent edge weights.

Discrete Harmonic One-form



for each boundary vertex v_i, we obtain an equation:

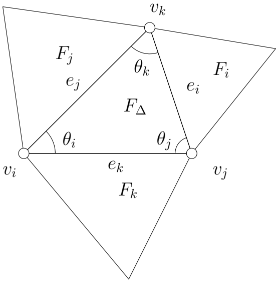
$$\sum_{j=0}^{n-1} w_{ij}\ \omega([v_i, v_j]) \boxed{-w_{i,n}\ \omega([v_n, v_i])} = \sum_{j=0}^{n} w_{ij}(\eta_j - \eta_i). \quad (9)$$

Algorithm for Random Harmonic One-form

Input:A closed genus one mesh M;
output: A basis of harmonic one-form group;

1. Generate a random one form ω, assign each $\omega(e)$ a random number;
2. Compute cotangent edge weight using Eqn. (1);
3. Compute the coexact form δF using Eqn. (6);
4. Compute the exact form df using Eqn. (8);
5. Harmonic 1-form is obtained by $h = \omega - d\eta - \delta\Omega$;

Wedge Product



Given two one-forms ω_1 and ω_2 on a triangle mesh M, then the 2-form $\omega_1 \wedge \omega_2$ on each face $\Delta = [v_i, v_j, v_k]$ is evaluated as

$$\omega_1 \wedge \omega_2(\Delta) = \frac{1}{6} \begin{vmatrix} \omega_1(e_i) & \omega_1(e_j) & \omega_1(e_k) \\ \omega_2(e_i) & \omega_2(e_j) & \omega_2(e_k) \\ 1 & 1 & 1 \end{vmatrix} \tag{10}$$

Wedge Product Formula

Proof.

Since ω_1 and ω_2 are linear,

$$\int_\Delta \omega_1 \wedge \omega_2 = \frac{1}{2}\omega_1 \wedge \omega_2(e_i \times e_j)$$

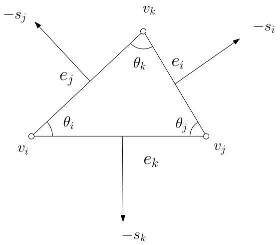
$$=\frac{1}{6}[\omega_1 \wedge \omega_2(e_i \times e_j) + \omega_1 \wedge \omega_2(e_j \times e_k) + \omega_1 \wedge \omega_2(e_k \times e_i)]$$

$$=\frac{1}{6}\left\{\begin{vmatrix} \omega_1(e_i) & \omega_1(e_j) \\ \omega_2(e_i) & \omega_2(e_j) \end{vmatrix} + \begin{vmatrix} \omega_1(e_j) & \omega_1(e_k) \\ \omega_2(e_j) & \omega_2(e_j) \end{vmatrix} + \begin{vmatrix} \omega_1(e_k) & \omega_1(e_i) \\ \omega_2(e_k) & \omega_2(e_i) \end{vmatrix}\right\}$$

$$=\frac{1}{6}\begin{vmatrix} \omega_1(e_i) & \omega_1(e_j) & \omega_1(e_k) \\ \omega_2(e_i) & \omega_2(e_j) & \omega_2(e_k) \\ 1 & 1 & 1 \end{vmatrix}$$

□

Wedge Product Formula



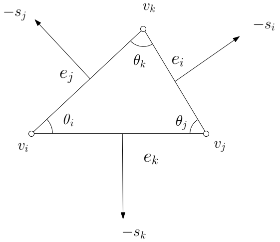
Set $f:\Delta\to\mathbb{R}$,

$$\begin{cases} f(v_i)=0\\ f(v_j)=\omega(e_k)\\ f(v_k)=-\omega(e_j)\end{cases}$$

$$\nabla f(p)=\frac{1}{2A}(f(v_i)\mathbf{s}_i+f(v_j)\mathbf{s}_j+f(v_k)\mathbf{s}_k)$$

$$\begin{aligned}\mathbf{w}&=\frac{1}{2A}[\omega(e_k)\mathbf{s}_j-\omega(e_j)\mathbf{s}_k]\\&=\frac{\mathbf{n}}{2A}\times[\omega(e_k)(\mathbf{v}_i-\mathbf{v}_k)-\omega(e_j)(\mathbf{v}_j-\mathbf{v}_i)]\\&=-\frac{\mathbf{n}}{2A}\times[\omega(e_k)\mathbf{v}_k+\omega(e_j)\mathbf{v}_j+\omega(e_i)\mathbf{v}_i]\end{aligned}$$

Wedge Product Formula



$$\int_\Delta \omega_1 \wedge \omega_2 = A|\mathbf{w}_1 \times \mathbf{w}_2|$$

$$= \frac{A}{4A^2}(\omega_k^1\omega_j^2 - \omega_j^1\omega_k^2)|\mathbf{s}_j \times \mathbf{s}_k|$$

$$= \frac{1}{2}\begin{vmatrix} \omega_k^1 & \omega_j^1 \\ \omega_k^2 & \omega_j^2 \end{vmatrix}$$

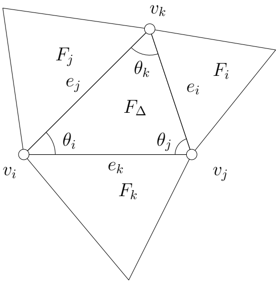
$$\mathbf{w} = \frac{1}{2A}(\omega_k \mathbf{s}_j - \omega_j \mathbf{s}_k)$$

since $\omega_i^\gamma + \omega_j^\gamma + \omega_k^\gamma = 0$, $\gamma = 1, 2$, we obtain

$$\mathbf{w} = \frac{-1}{6A}\begin{vmatrix} \omega_i & \omega_j & \omega_k \\ \mathbf{s}_i & \mathbf{s}_j & \mathbf{s}_k \\ 1 & 1 & 1 \end{vmatrix}$$

$$\boxed{\int_\Delta \omega_1 \wedge \omega_2 = \frac{1}{6}\begin{vmatrix} \omega_k^1 & \omega_j^1 & \omega_i^1 \\ \omega_k^2 & \omega_j^2 & \omega_i^2 \\ 1 & 1 & 1 \end{vmatrix}}$$

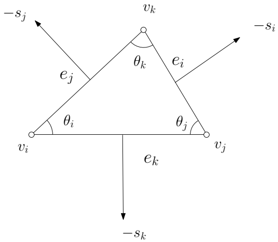
Wedge Product



Given two one-forms ω_1 and ω_2 on a triangle mesh M, then the 2-form $\omega_1 \wedge {}^*\omega_2$ on each face $\Delta = [v_i, v_j, v_k]$ is evaluated as

$$\omega_1 \wedge {}^*\omega_2(\Delta) = \frac{1}{2}[\cot\theta_i \omega_1(e_i)\omega_2(e_i) + \cot\theta_j \omega_1(e_j)\omega_2(e_j) + \cot\theta_k \omega_1(e_k)\omega_2(e_k)] \tag{11}$$

Wedge Product Formula



$$w_1 = \frac{1}{2A}(\omega_k^1 s_j - \omega_j^1 s_k)$$

$$w_2 = \frac{1}{2A}(\omega_k^2 s_j - \omega_j^2 s_k)$$

$$\int_\Delta \omega_1 \wedge {}^*\omega_2 = A\langle w_1, w_2\rangle$$

$$= \frac{1}{4A}\left\{\omega_k^1\omega_k^2\langle s_j, s_j\rangle + \omega_j^1\omega_j^2\langle s_k, s_k\rangle\right.$$

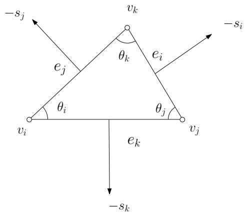
$$\left.-(\omega_k^1\omega_j^2 + \omega_j^1\omega_k^2)\langle s_j, s_k\rangle\right\}$$

$$= \frac{1}{4A}\left\{-\omega_k^1\omega_k^2\langle s_j, s_i + s_k\rangle\right.$$

$$-\omega_j^1\omega_j^2\langle s_k, s_i + s_j\rangle$$

$$\left.-(\omega_k^1\omega_j^2 + \omega_j^1\omega_k^2)\langle s_j, s_k\rangle\right\}$$

Wedge Product Formula



$$
\begin{aligned}
&= \frac{1}{4A}\left\{-\omega_k^1\omega_k^2\langle s_j, s_i\rangle - \omega_k^1\omega_k^2\langle s_j, s_k\rangle\right.\\
&- \omega_j^1\omega_j^2\langle s_k, s_i\rangle - \omega_j^1\omega_j^2\langle s_k, s_j\rangle\\
&\left.-(\omega_k^1\omega_j^2 + \omega_j^1\omega_k^2)\langle s_j, s_k\rangle\right\}\\
&= -\omega_k^1\omega_k^2\frac{\langle s_j, s_i\rangle}{4A} - \omega_j^1\omega_j^2\frac{\langle s_k, s_i\rangle}{4A}\\
&- \frac{\langle s_k, s_j\rangle}{4A}(\omega_k^1\omega_k^2 + \omega_j^1\omega_j^2 + \omega_k^1\omega_j^2 + \omega_j^1\omega_k^2)\\
&= -\omega_k^1\omega_k^2\frac{\langle s_j, s_i\rangle}{4A} - \omega_j^1\omega_j^2\frac{\langle s_k, s_i\rangle}{4A}\\
&- \frac{\langle s_k, s_j\rangle}{4A}(\omega_k^1 + \omega_j^1)(\omega_k^2 + \omega_j^2)\\
&= -\omega_k^1\omega_k^2\frac{\langle s_j, s_i\rangle}{4A} - \omega_j^1\omega_j^2\frac{\langle s_k, s_i\rangle}{4A} - \omega_i^1\omega_i^2\frac{\langle s_j, s_k\rangle}{4A}\\
&= \frac{1}{2}\left(\omega_i^1\omega_i^2\cot\theta_i + \omega_j^1\omega_j^2\cot\theta_j + \omega_k^1\omega_k^2\cot\theta_k\right)
\end{aligned}
$$

Holomorphic 1-form Basis

Given a set of harmonic 1-form basis $\omega_1, \omega_2, \dots, \omega_{2g}$; in smooth case, the conjugate 1-form ${}^*\omega_i$ is also harmonic, therefore

$$
{}^*\omega_i = \lambda_{i1}\omega_1 + \lambda_{i2}\omega_2 + \cdots + \lambda_{i,2g}\omega_{2g},
$$

We get linear equation group,

$$
\begin{pmatrix} \omega_1 \wedge {}^*\omega_i \\ \omega_2 \wedge {}^*\omega_i \\ \vdots \\ \omega_{2g} \wedge {}^*\omega_i \end{pmatrix} = \begin{pmatrix} \omega_1 \wedge \omega_1 & \omega_1 \wedge \omega_2 & \cdots & \omega_1 \wedge \omega_{2g} \\ \omega_2 \wedge \omega_1 & \omega_2 \wedge \omega_2 & \cdots & \omega_2 \wedge \omega_{2g} \\ \vdots & \vdots & & \vdots \\ \omega_{2g} \wedge \omega_1 & \omega_{2g} \wedge \omega_2 & \cdots & \omega_{2g} \wedge \omega_{2g} \end{pmatrix} \begin{pmatrix} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,2g} \end{pmatrix} \tag{12}
$$

We take the integration of each element on both left and right side, and solve the λ_{ij}'s.

Holomorphic 1-form Basis

In order to reduce the random error, we integrate on the whole mesh,

$$\begin{pmatrix} \int_M \omega_1 \wedge {}^*\omega_i \\ \int_M \omega_2 \wedge {}^*\omega_i \\ \vdots \\ \int_M \omega_{2g} \wedge {}^*\omega_i \end{pmatrix} = \begin{pmatrix} \int_M \omega_1 \wedge \omega_1 & \cdots & \int_M \omega_1 \wedge \omega_{2g} \\ \int_M \omega_2 \wedge \omega_1 & \cdots & \int_M \omega_2 \wedge \omega_{2g} \\ \vdots & & \vdots \\ \int_M \omega_{2g} \wedge \omega_1 & \cdots & \int_M \omega_{2g} \wedge \omega_{2g} \end{pmatrix} \begin{pmatrix} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,2g} \end{pmatrix} \tag{13}$$

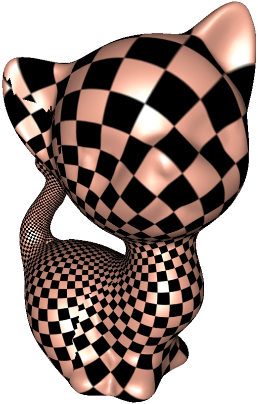
and solve the linear system to obtain the coefficients.

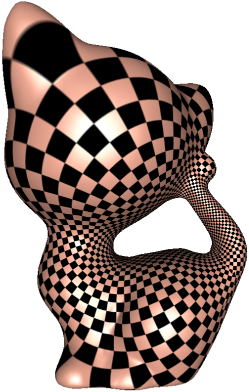
Algorithm for Holomorphic 1-form Basis

Input: A set of harmonic 1-form basis $\omega_1, \omega_2, \dots, \omega_{2g}$;
Output: A set of holomorphic 1-form basis $\omega_1, \omega_2, \dots, \omega_{2g}$;

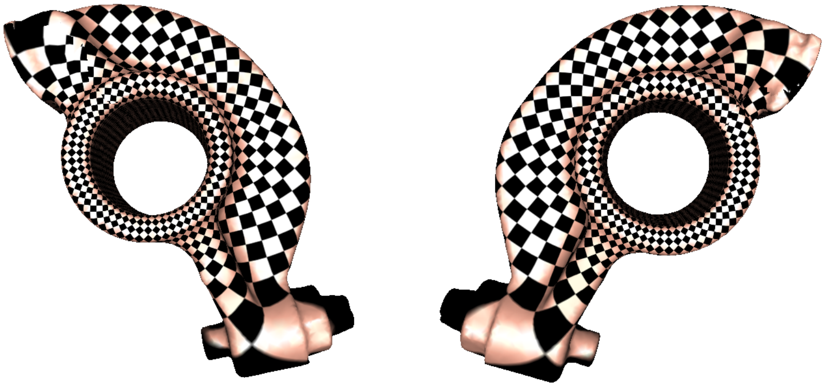
1. Compute the integration of the wedge of ω_i and ω_j, $\int_M \omega \wedge \omega_j$, using Eqn. (10);
2. Compute the integration of the wedge of ω_i and ${}^*\omega_j$, $\int_M \omega \wedge {}^*\omega_j$, using Eqn. (11);
3. Solve linear equation group Eqn. (13), obtain the linear combination coefficients, get conjugate harmonic 1-forms, ${}^*\omega_i = \sum_{j=1}^{2g} \lambda_{ij}\omega_j$
4. Form the holomorphic 1-form basis $\{\omega_i + \sqrt{-1}{}^*\omega_i, \quad i = 1, 2, \dots, 2g\}$.

Holomorphic One-form

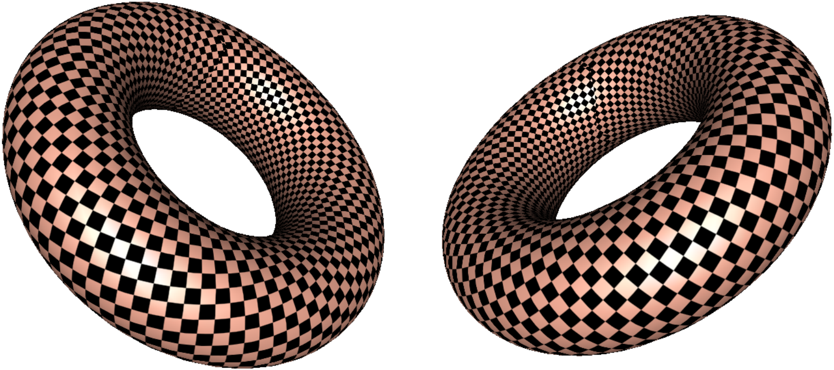




Holomorphic One-form



Holomorphic One-form



Cross Fields on Surfaces

Index Theorem

Theorem (Cross Field Singularity)

Suppose $(S, \mathbf{g})$ *is an orientable, closed metric surface. Given a 0-form* $\theta = \sum_{i=1}^{n} \lambda_i p_i$*, where* $\lambda_i \in \mathbb{Z}$*,* $\lambda_i \leq 2$*, then* θ *is the singularity configuration of a continuous cross field on* S*, if and only if*

$$\sum_{i=1}^{n} \lambda_i = 4\chi(S), \tag{14}$$

where $\chi(S)$ *is the Euler characteristic number of the surface,* λ_i *is the index of* p_i*.*

Cross Field Construction Algorithm

Input : Closed Triangle mesh M, singularities $\theta = \sum_i \lambda_i p_i$
Output: Cross field σ with prescribed singularities θ

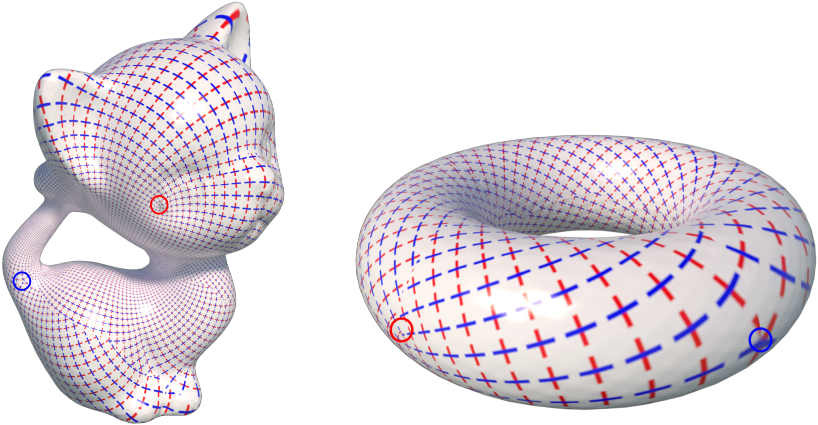
1. Set target curvature $\bar{K}_i = \lambda_i \pi/2$;
2. Compute a flat metric $\bar{\mathbf{g}}$ with target curvature using Ricci flow;
3. Choose a base point $q \in S \setminus \{p_i\}$, compute the generators of fundamental group $\{\gamma_1, \gamma_2, \dots, \gamma_{2g}\}$;
4. Parallel transport a fixed cross c at the base point q along γ_i's to compute the holonomy β_k;
5. Compute harmonic 1-form basis of $H^1_{dR}(M, \mathbb{R})$ $\{\omega_1, \omega_2, \dots, \omega_{2g-1}, \omega_{2g}\}$, such that $\int_{\gamma_i} \omega_j = \delta_i^j$;
6. Construct a harmonic 1-form $\omega = \sum_i \beta_i \omega_i$;

Cross Field Construction Algorithm

For each vertex $v_i \in M$

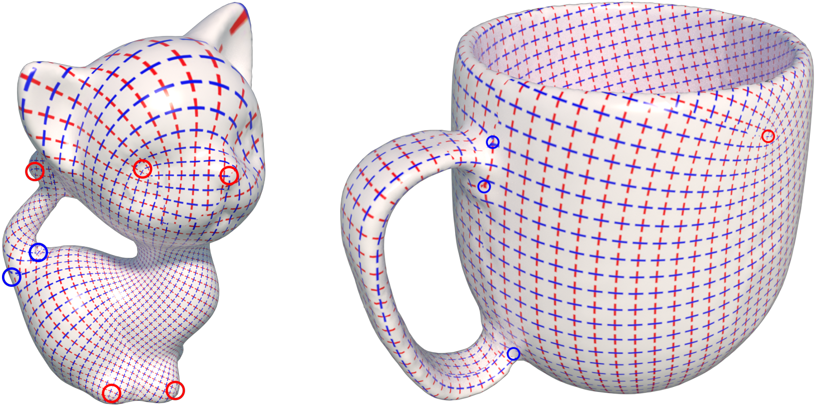
1. Find a path $\gamma \subset S \setminus \{q_i\}$ from q to v_i;
2. Parallel transport c along γ to obtain c';
3. Rotate c' by angle $\int_\gamma \omega$ clockwisely;

Singularities on a Topological Torus



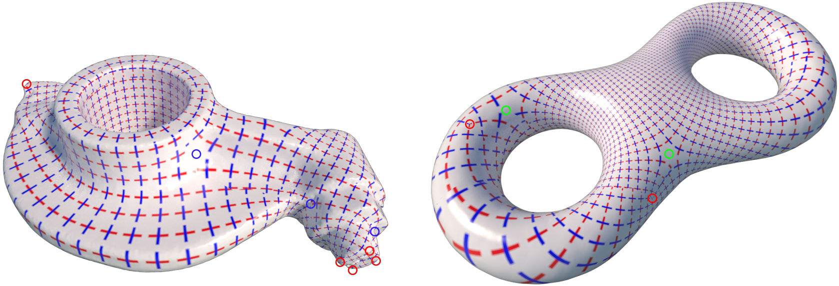
Smooth cross fields on genus one closed surfaces with two singularities.

Singularities on a Topological Torus



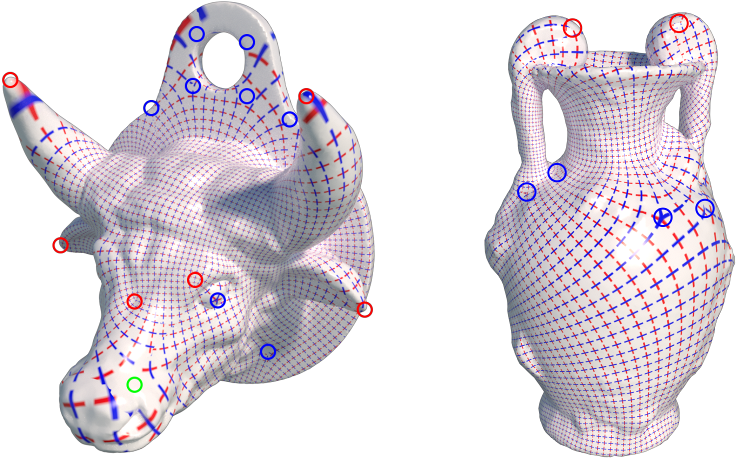
Smooth cross fields on genus one closed surfaces with two singularities.

Singularities on a Topological Torus



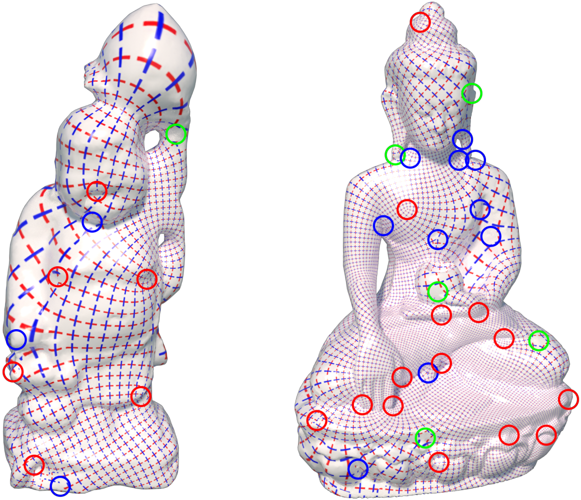
Smooth cross fields on surfaces.

Singularities on a Topological Torus



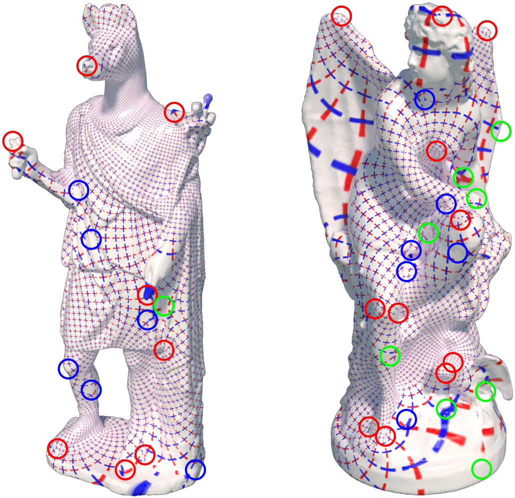
Smooth cross fields on surfaces.

Singularities on a Topological Torus



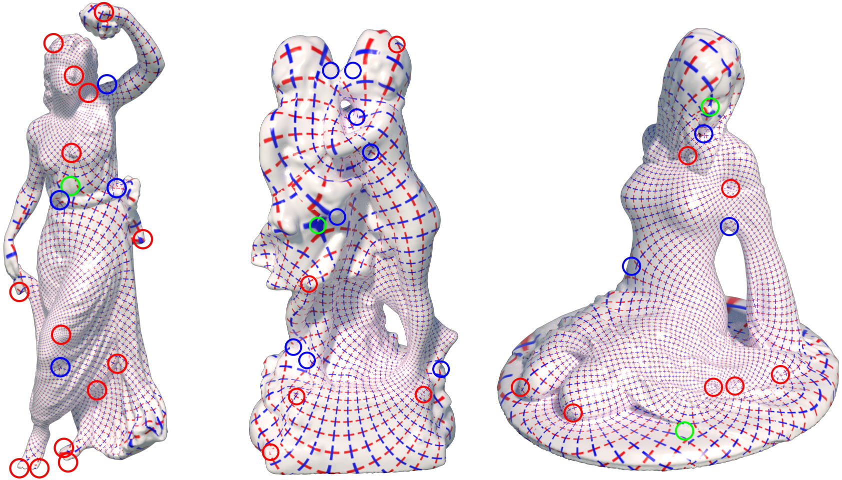
Smooth cross fields on surfaces.

Singularities on a Topological Torus



Smooth cross fields on surfaces.

Singularities on a Topological Torus



Smooth cross fields on surfaces.

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- Xianfeng Gu and Shing-Tung Yau. **Global Conformal Surface Parameterization**. *First Eurographics Symposium on Geometry Processing (SGP03)*, Pages:127-137, Aachen, Germany, June 23-25, 2003.
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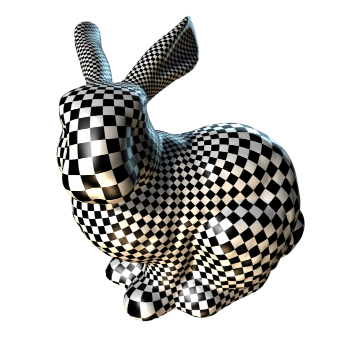
Thanks

For more information, please email to gu@cs.stonybrook.edu.



Thank you!