Differential Geometry for Mesh Generation I

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Unstructured surface mesh generation

Surface Mesh Generation

Motivation for Meshing

Surface mesh generation plays a fundamental role in many engineering and medical fields, specially CAD, CAE and CAM fields. Despite tens of years of intensive research, there are still remain many challenges.

Central Challenges

1. How to generate high quality meshes on surfaces with complicated topologies and geometric features?
2. How to generate anisotropic meshes?
3. How to generate structured quadrilateral meshes (hexahedral meshes for solids)?

Planar Mesh Generation

Planar Mesh Generation

Mesh generation on planar domain is relatively mature. There are many existing algorithms can produce good quality meshes, such as Delaunay refinement algorithm, Chew's second algorithm (30°), Ruppert's algorithm (20.7°), Centroidal Voronoi Tessellation algorithm and so on. These algorithms can guarantee the minimal angle has specific lower bounds.

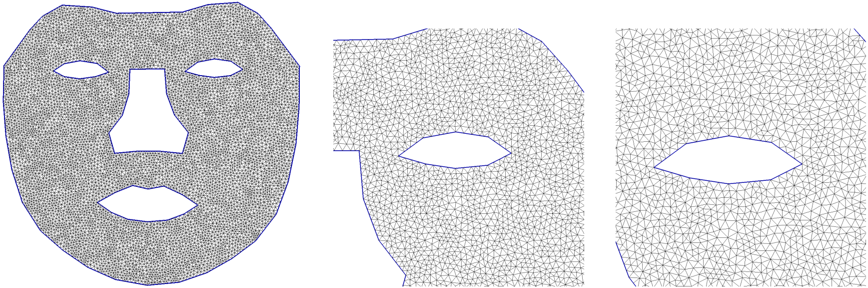


Figure: Ruppert's Delaunay refinement algorithm.

Surface Mesh Generation

Surface Mesh Generation

Surfaces generation is much more difficult due to their complicated topologies and geometries. There are still many theoretic problems, open for tens of years.

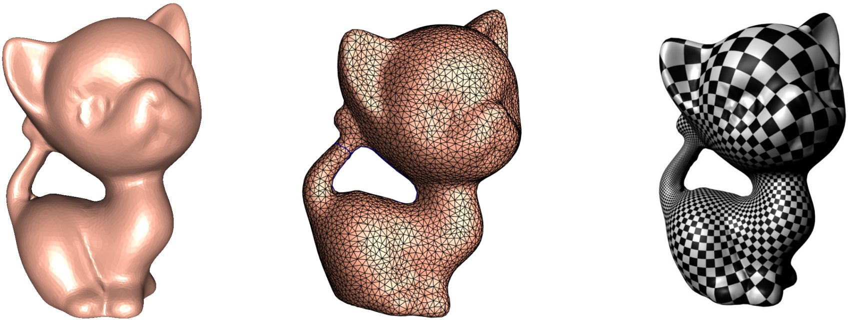


Figure: Meshing for a kitten model.

Surface Mesh Generation - Key Idea

Problem (Surface Meshing)

How to generate high quality triangle meshes on surfaces with complicated topology and geometry ?

Key Idea

Find a special diffeomorphism $\varphi : (S, \mathbf{g}) \to \Omega$ maps the 3D surface onto a planar domain, and converts the 3D meshing problem to a 2D planar meshing problem.

Surface Mesh Generation - Key Idea

Key Idea

Find a special diffeomorphism $\varphi : (S, \mathbf{g}) \to \Omega$ maps the 3D surface onto a planar domain, and converts the 3D meshing problem to a 2D planar meshing problem.

Figure: 3D meshing problems are converted to 2D meshing ones.

Special Mappings

- Angle preserving maps: keep the minimal angles;
- Area preserving maps: keep the grading;
- impossible to keep both, otherwise it is isometric.

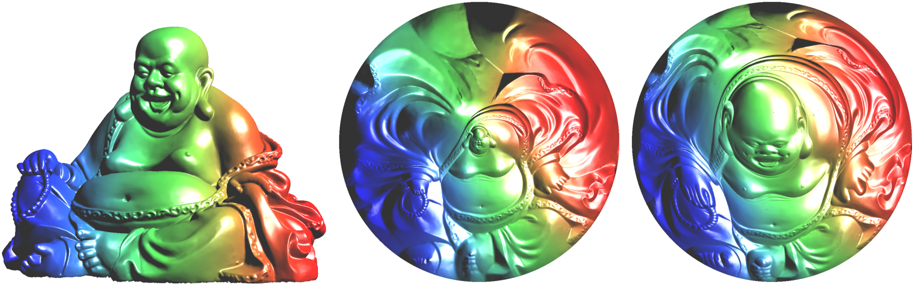


Figure: Conformal and optimal transport maps.

Special Mappings

Special Mappings

- Both can not be solved using conventional FEM;
- Both can be solved using geometric variational methods;
- Both do not require good initial meshes.

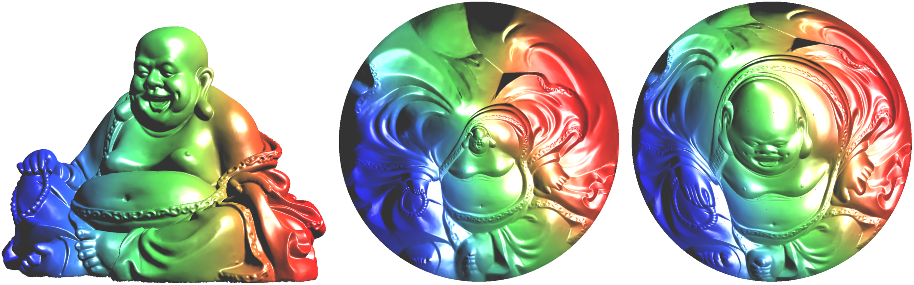


Figure: Conformal and optimal transport maps.

Conformal Mapping

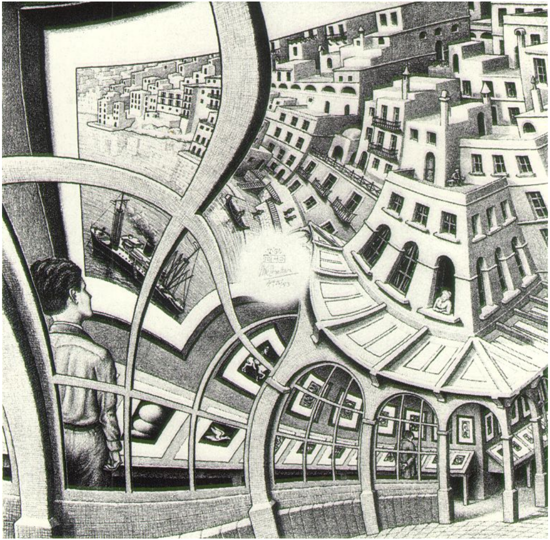


Figure: Gallery (Escher).

Conformal Mapping

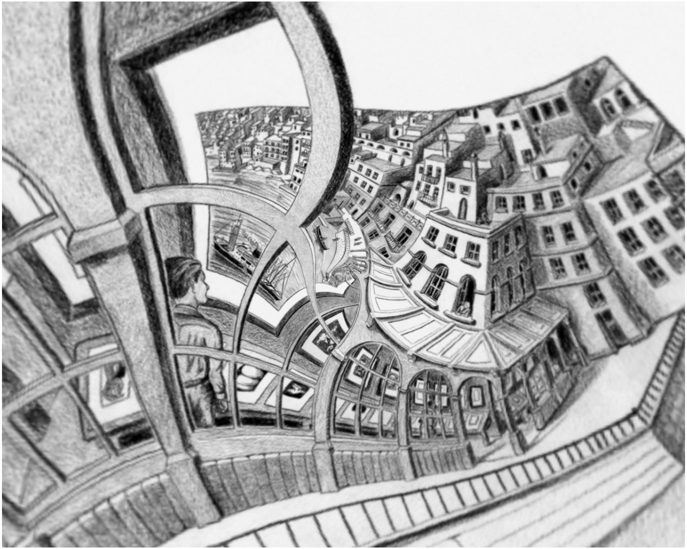


Figure: Gallery (Escher).

Conformal Mapping

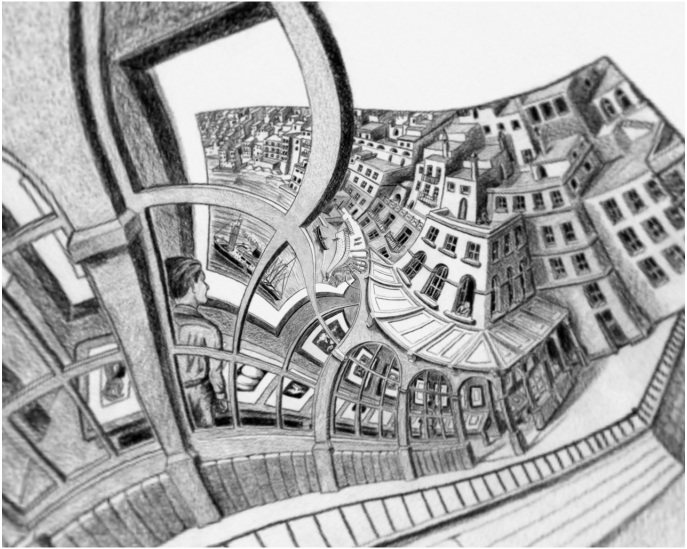


Figure: Gallery (Escher).

Conformal Mapping

Figure: Planar conformal map, local shape preserving.

Biholomorphic Function

Definition (holomorphic Function)

Suppose a complex function $f : \mathbb{C} \to \mathbb{C}$, $f : z \mapsto w$ is holomorphic, if it satisfies the Cauchy-Riemann equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

where $z = x + iy$ and $w = u + iv$. If f is inevitable, and f^{-1} is also holomorphic, then f is biholomorphic.

Planar conformal maps are biholomorphic functions.

Conformal Mapping

Figure: Planar conformal map, local shape preserving.

Conformal Mapping

Definition (Conformal Mapping)

Suppose $\varphi : (S, \mathbf{g}) \to (T, \mathbf{h})$ is a C^1 mapping, if the pull back metric $\varphi^*\mathbf{h}$ satisfies the condition

$$\varphi^*\mathbf{h} = e^{2\lambda}\mathbf{g},$$

where $\lambda : S \to \mathbb{R}$ is the conformal factor, then φ is a conformal map.

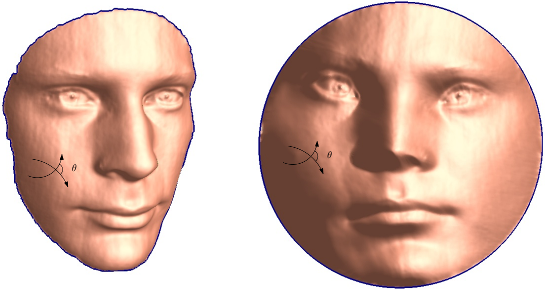


Figure: Angle preserving property.

Conformal Mapping

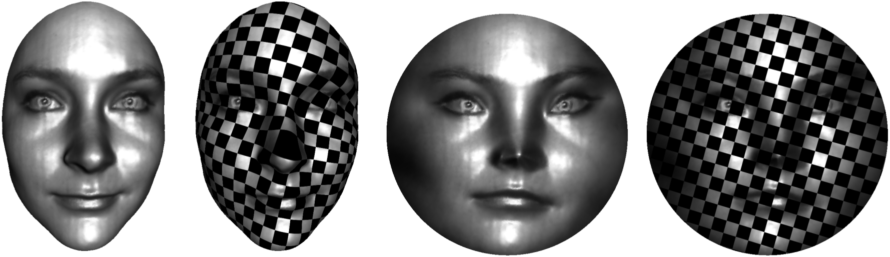


Figure: Angle preserving property.

Disk Harmonic Maps

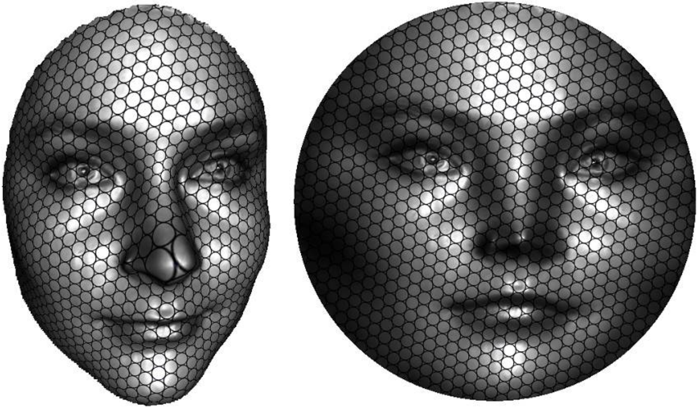


Figure: Harmonic map between topological disks.

Conformal Mapping

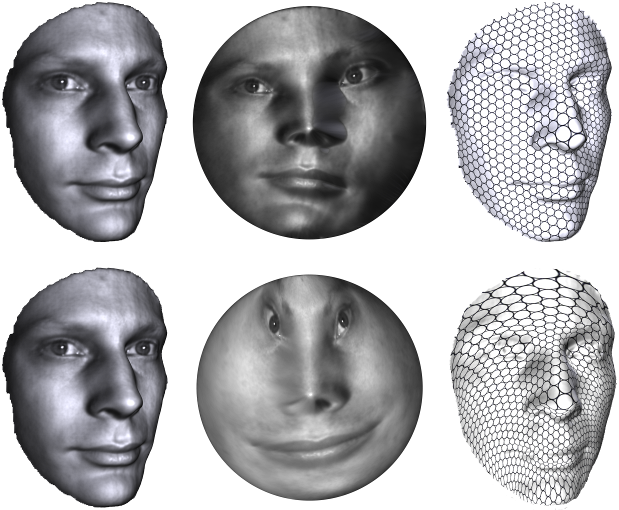


Figure: Infinitesimal circle preserving property.

Discrete Surface Ricci Flow

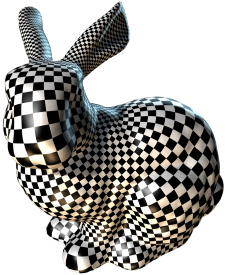
Isothermal Coordinates

Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u, v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

Gaussian Curvature

Under the isothermal coordinates, the Riemannian metric is $\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2)$, then the Gaussian curvature on interior points are

$$K = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda}}\Delta\lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

Conformal Metric Deformation

Definition

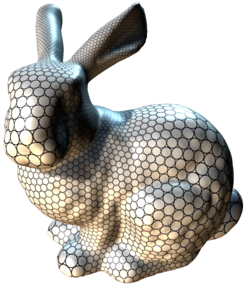
Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Yamabi Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(-\partial_n\lambda + k_g).$$

Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.

Surface Uniformization

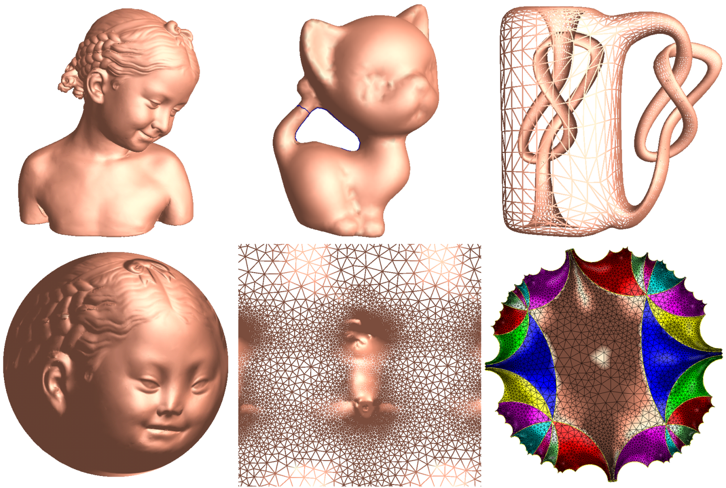


Figure: Closed surface uniformization.

Surface Uniformization

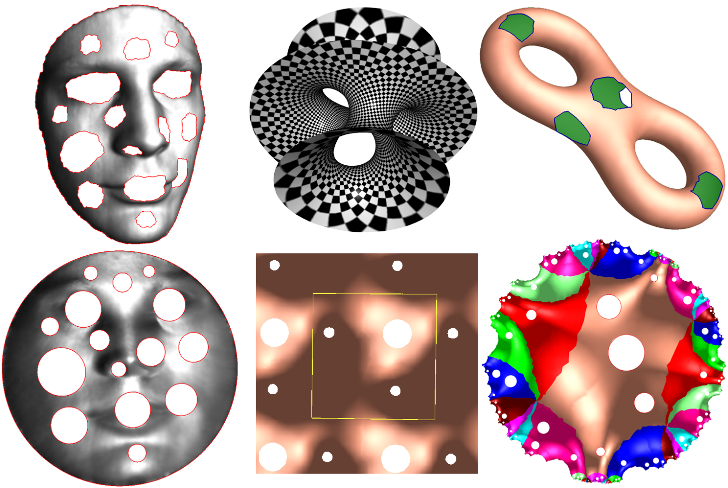


Figure: Open surface uniformization.

Surface Ricci Flow

Proposition

During the curvature flow $\frac{d\lambda}{dt} = -K$, then

$$\frac{d}{dt}K = 2K^2 + \Delta_{\mathbf{g}} K.$$

$$\begin{aligned}
\frac{d}{dt}K &= \frac{d}{dt}(-e^{-2\lambda}\Delta\lambda) \\
&= -\left(-2\frac{d\lambda}{dt}\right) e^{-2\lambda}\Delta\lambda - e^{-2\lambda}\Delta\frac{d\lambda}{dt} \\
&= \left(-2\frac{d\lambda}{dt}\right) \boxed{-e^{-2\lambda}\Delta\lambda} - \boxed{e^{-2\lambda}\Delta}\frac{d\lambda}{dt} \\
&= \left(-2\frac{d\lambda}{dt}\right) K - \Delta_{\mathbf{g}}\frac{d\lambda}{dt} \\
&= 2K^2 + \Delta_{\mathbf{g}} K
\end{aligned}$$

Surface Ricci Flow

Key Idea

$$K = -\Delta_{\mathbf{g}}\lambda,$$

Roughly speaking,

$$\frac{dK}{dt} = \frac{d}{dt}\Delta_{\mathbf{g}}\lambda$$

Let $\frac{d\lambda}{dt} = -K$,

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K + 2K^2$$

Diffusion and reaction equation!

Surface Ricci Flow

Definition (Hamilton's Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -2Kg_{ij}.$$

The normalized surface Ricci flow,

$$\frac{dg_{ij}}{dt} = \frac{2\pi\chi(S)}{A(0)} - 2Kg_{ij},$$

where $A(0)$ is the initial surface area.

The normalized surface Ricci flow is area-preserving, the Ricci flow will converge to a metric such that the Gaussian curvature is constant $\frac{2\pi\chi(S)}{A(0)}$ every where.

Ricci Flow

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Summary

Surface Ricci Flow

- Conformal metric deformation

$$\mathbf{g} \to e^{2u}\mathbf{g}$$

- Curvature Change - heat diffusion

$$\frac{dK}{dt} = \Delta_{\mathbf{g}} K + 2K^2$$

- Ricci flow

$$\frac{du}{dt} = \bar{K} - K.$$

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.

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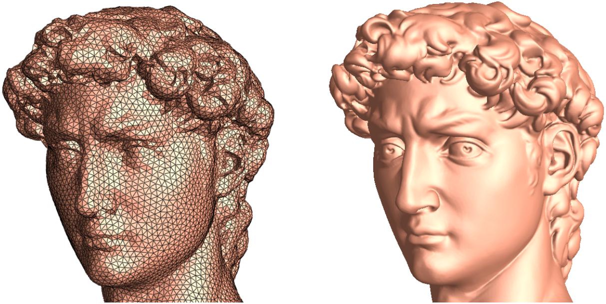
Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.



Discrete Generalization

Concepts

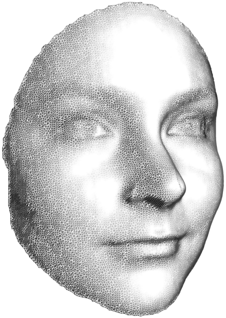
1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation

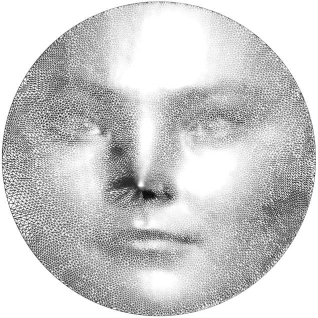
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

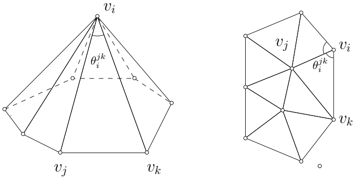
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \rightarrow \mathbb{R}^1$.

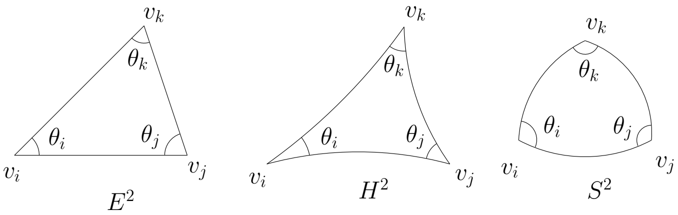
$$K(v_i) = 2\pi - \sum_{jk} \theta_i^{jk}, v_i \notin \partial M; K(v_i) = \pi - \sum_{jk} \theta_{jk}, v_i \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics Determines the Curvatures



cosine laws

$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \qquad \mathbb{S}^2$$

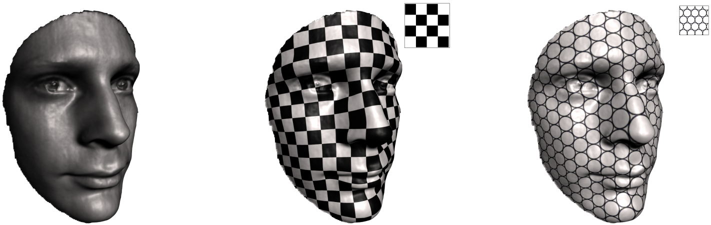
$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} \qquad \mathbb{H}^2$$

$$1 = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \qquad \mathbb{E}^2$$

Discrete Conformal Metric Deformation

Conformal maps Properties

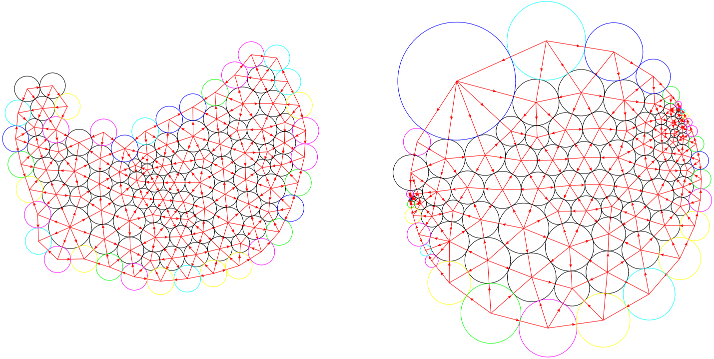
- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.



Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

Discrete Conformal Metric Deformation vs CP



Circle Packing Metric

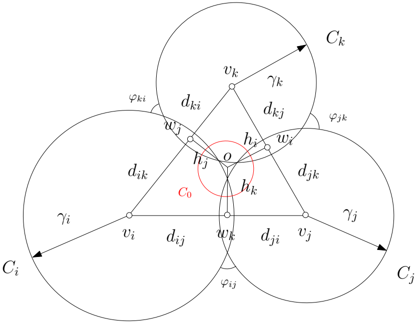
CP Metric

We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\varphi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\varphi_{ij} | \forall e_{ij}\}$$



Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u} : V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation

$$d\mathbf{K} = \Delta d\mathbf{u},$$

Δ is a discrete Lapalce-Beltrami operator.

Unified Framework of Discrete Curvature Flow

Analogy

- Curvature flow
$$\frac{du}{dt} = \bar{K} - K,$$
- Energy
$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$
- Hessian of E denoted as Δ,
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

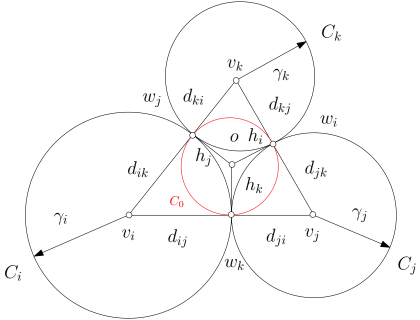
Criteria for Discretization

Key Points

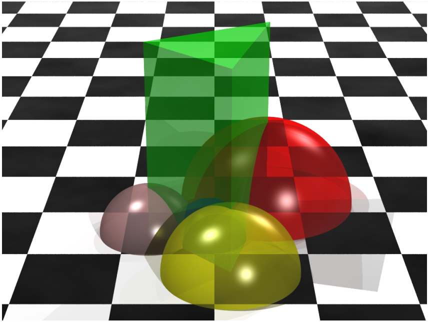
- Convexity of the energy $E(\mathbf{u})$
- Convexity of the metric space ($\mathbf{u}$-space)
- Admissible curvature space ($\mathbf{K}$-space)
- Preserving or reflecting richer structures
- Conformality

Unified Discrete Surface Ricci Flow

Unified Ricci Flow



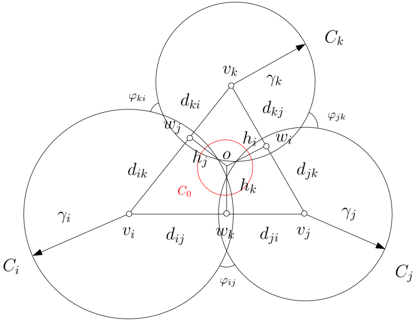
(a)Tangential CP



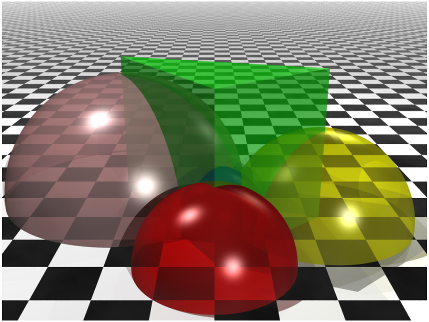
(b) Generalized Hyperbolic Tetrahedron, $(\eta, \epsilon) = (1, 1)$

Figure: Tangential circle packing.

Thurston's Circle Packing



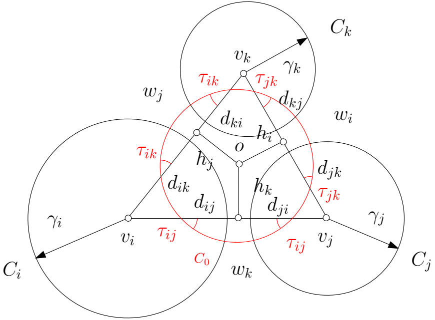
(a)Thurston's Circle packing



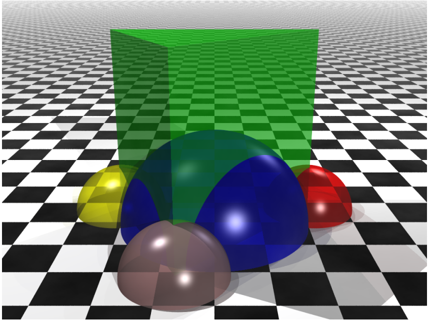
(b)Generalized Hyperbolic Tetrahedron, $0 \leq \eta < 1, \epsilon = 1$

Figure: Thurston's circle packing.

Inversive Distance Circle Packing



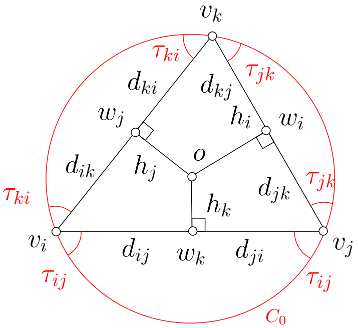
(c)Inversive distance CP



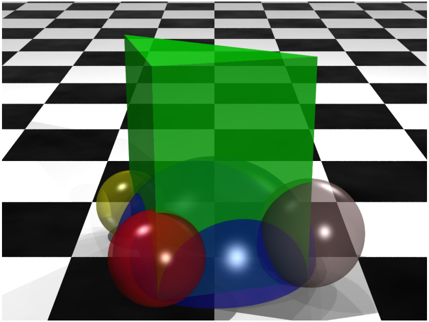
(d)Generalized Hyperbolic Tetrahedron, $\eta > 1, \epsilon = 1$

Figure: Inversive distance circle packing.

Yamabe Flow



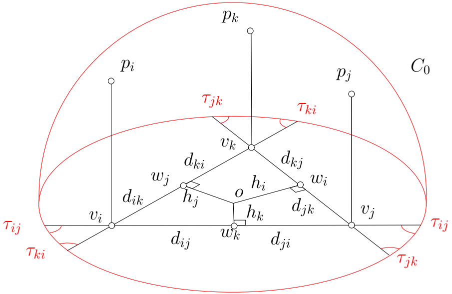
(d)Yamabe flow



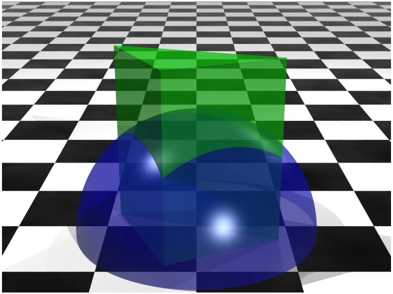
(e)Generalized Hyperbolic Tetrahedron, $\eta > 0, \epsilon = 0$

Figure: Yamabe flow.

Virtual Radius Circle Packing



(e)Virtual radius CP

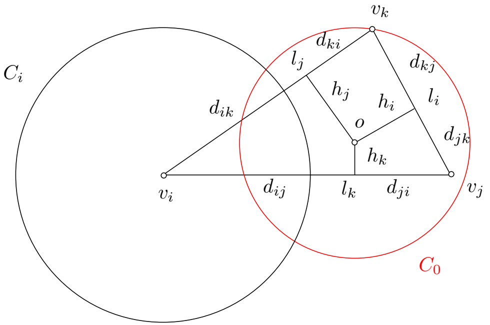


(f)Generalized Hyperbolic Tetrahedron, $\eta > 0, \epsilon = -1$

Figure: virtual radius circle packing.

$$l_k^2 = -r_i^2 - r_j^2 + 2\eta_{ij} r_i r_j.$$

Mixed Type



(f)mixed type

Figure: Mixed typed circle packing.

Conformal Factor

Definition (Discrete Conformal Factor)

The discrete conformal factor is defined as $u : V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{E}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Edge Length

Definition (Edge Length)

The edge lengths are given by

$$u_i = \begin{cases} l_{ij}^2 &= 2\eta_{ij} e^{u_i+u_j} + \varepsilon_i e^{2u_i} + \varepsilon_j e^{2u_j} & \mathbb{E}^2 \\ \cosh l_{ij} &= \frac{4\eta_{ij} e^{u_i+u_j} + (1+\varepsilon_i e^{2u_i})(1+\varepsilon_j e^{2u_j})}{(1-\varepsilon_i e^{2u_i})(1-\varepsilon_j e^{2u_j})} & \mathbb{H}^2 \\ \cos l_{ij} &= \frac{-4\eta_{ij} e^{u_i+u_j} + (1-\varepsilon_i e^{2u_i})(1-\varepsilon_j e^{2u_j})}{(1+\varepsilon_i e^{2u_i})(1+\varepsilon_j e^{2u_j})} & \mathbb{S}^2 \end{cases}$$

Edge Length

Scheme	ε_i	ε_j	η_{ij}
Tangential Circle Packing	$+1$	$+1$	$+1$
Thurston's Circle Packing	$+1$	$+1$	$[0,1]$
Inversive Distance Circle Packing	$+1$	$+1$	$(0,\infty)$
Yamabe Flow	0	0	$(0,\infty)$
Virtual Distance Circle Packing	-1	-1	$(0,\infty)$
Mixed Type	$\{-1,0,+1\}$	$\{-1,0,+1\}$	$(0,\infty)$

Table: Parameters for schemes.

Entropy Energy

Definition (Entroy on a Face)

A discrete surface with $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$ background geometry, and a circle packing metric $(\Sigma, \gamma, \eta, \varepsilon)$. For each triangle $[v_i, v_j, v_k]$ with inner angle $(\theta_i, \theta_j, \theta_k)$, the entropy energy for the face is given by

$$E_f(u_i, u_j, u_k) = \int^{(u_i, u_j, u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k.$$

Entropy Energy

Definition (Entroy on a mesh)

A discrete surface with $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$ background geometry, and a circle packing metric $(\Sigma, \gamma, \eta, \varepsilon)$. The discrete entropy energy for the whole mesh is defined as

$$E_= \int^{(u_1,u_2,\cdots,u_n)} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i.$$

The mesh entropy can be represented as the face energies

$$E_\sigma = \sum_{i=1}^{n} (\bar{K}_i - 2\pi) u_i + \sum_{f \in F} E_f.$$

Symmetry

Suppose a triangle $[v_i, v_j, v_k]$ is with background geometry $\mathbb{S}^2$,$\mathbb{E}^2$, $\mathbb{H}^2$, conformal factor (u_i, u_j, u_k), edge length (l_i, l_j, l_k), inner angles $(\theta_i, \theta_j, \theta_k)$, entropy energy is

$$E(u_i, u_j, u_k) = \int^{(u_i,u_j,u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k. \tag{1}$$

Then the Hessian matrix is given by

$$\frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} = -\frac{1}{2A} L\Theta L^{-1} D, \tag{2}$$

where, A is the triangle area

$$A = \frac{1}{2} \sin\theta_i s(l_j) s(l_k), \tag{3}$$

Symmetry

The matrix L is

$$L = \begin{pmatrix} s(l_i) & 0 & 0 \\ 0 & s(l_j) & 0 \\ 0 & 0 & s(l_k) \end{pmatrix} \tag{4}$$

Θ

$$\Theta = \begin{pmatrix} -1 & \cos\theta_k & \cos\theta_j \\ \cos\theta_k & -1 & \cos\theta_i \\ \cos\theta_j & \cos\theta_i & -1 \end{pmatrix} \tag{5}$$

matrix D is

$$D = \begin{pmatrix} 0 & \tau(i,j,k) & \tau(i,k,j) \\ \tau(j,i,k) & 0 & \tau(j,k,i) \\ \tau(k,i,j) & \tau(k,j,i) & 0 \end{pmatrix} \tag{6}$$

Hessian Matrix

where

$$s(x) = \begin{cases} x & \mathbb{E}^2 \\ \sinh x & \mathbb{H}^2 \\ \sin x & \mathbb{S}^2 \end{cases}$$

and

$$\tau(i,j,k) = \begin{cases} \frac{1}{2}(l_i^2 + \epsilon_j \gamma_j^2 - \epsilon_k \gamma_k^2) & \mathbb{E}^2 \\ \cosh l_i \cosh^{\epsilon_j} \gamma_j - \cosh^{\epsilon_k} \gamma_k & \mathbb{H}^2 \\ \cos l_i \cos^{\epsilon_j} \gamma_j - \cos^{\epsilon_k} \gamma_k & \mathbb{S}^2 \end{cases}$$

Geometric Interpretation

For each triangle, there is a power circle, orthogonal to three vertex circles. The distance from the power center to each edge is h_i, h_j, h_k. Then we have the geometric interpretation to the Hessian matrix: with $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$ background geometry,

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{h_3}{l_3}$$

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{\tanh h_3}{\sinh^2 l_3}\sqrt{2\cosh^{\epsilon_1} r_1 \cosh^{\varepsilon_2} r_2 \cosh l_3 - \cosh^{2\varepsilon_1} r_1 - \cosh^{2\varepsilon_2} r_2}$$

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{\tan h_3}{\sin^2 l_3}\sqrt{-2\cos^{\varepsilon_1} r_1 \cos^{\varepsilon_2} r_2 \cos l_3 + \cos^{2\varepsilon_1} r_1 + \cos^{2\varepsilon_2} r_2}$$

Existence and Uniqueness Theorem for Discrete Ricci Flow

Discrete Conformality

Definition (Discrete Conformality)

Two discrete metrics d, d' on (S, V) are discrete conformal if there exists sequence of discrete metrics on (S, V), $d = d_1, d_2, \ldots, d_m = d'$, and triangulations of (S, V), $T_1, T_2, \ldots, T_m$ satisfying

1. each T_i is Delaunay in d_i;
2. if $T_i = T_{i+1}$, there exists a discrete conformal factor $u : V \to \mathbb{R}$, for each edge $e \in T_i$ with vertices v_1 and v_2, then
$$l_{d_i}(e) = l_{d_i}(e)e^{u(v_1)+u(v_2)},$$
3. if $T_i \neq T_{i+1}$, then (S, d_i) is isometric to (S, d_{i+1}) by an isometry homotopic to the identity in (S, V).

The discrete conformal class of discrete metrics is called a discrete Riemann surface.

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The discrete conformal class of discrete metrics is called a discrete Riemann surface.

Discrete Uniformization Theorem

Theorem (Existence and Uniqueness)

Suppose (S, V) is a closed connected marked surface and d is any discrete metric on (S, V). The for any discrete Gaussian curvature $K^* : V \to (-\infty, 2\pi)$ with $\sum_{v \in V} K^*(v) = 2\pi\chi(S)$, there exists a discrete metric d', unique up to scaling on (S, V), so that d' is discrete conformal to d and the discrete curvature of d' is K^*. Furthermore, the discrete curvature flow with surgery associated to curvature K^* with initial value d converges to d' exponentially fast.

X. Gu, F. Luo, J. Sun and T. Wu, "A Discrete Uniformization Theorem for Polyhedral Surfaces", Journal of Differential Geometry, Volume 109, Number 2, Pages 223-256, 2018.

Discrete Uniformization Theorem

Definition (Discrete Entropy Energy)

The entropy energy of (S, V, d) is defined as

$$E(u) := \int^{(u_1, u_2, \dots, u_n)} \sum_{v_i \in V} (\bar{K}(v_i) - K(v_i)) du_i.$$

The discrete Ricci flow is the gradient flow of the entropy energy:

$$\frac{du_i(t)}{dt} = \bar{K}(v_i) - K(v_i, t),$$

the entropy is strictly concave on the space $\sum_i u_i = 0$, therefore can be optimized using Newton's method.

Experimental Results

Robustness Test

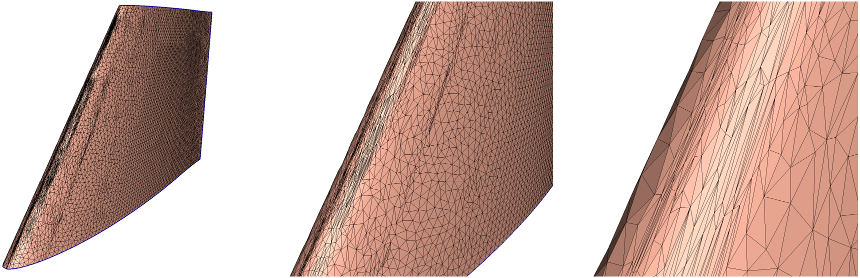


Figure: Initial mesh is with low quality triangulation.

Robustness Test

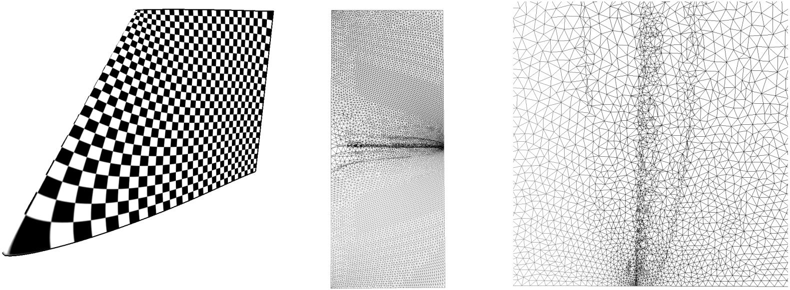
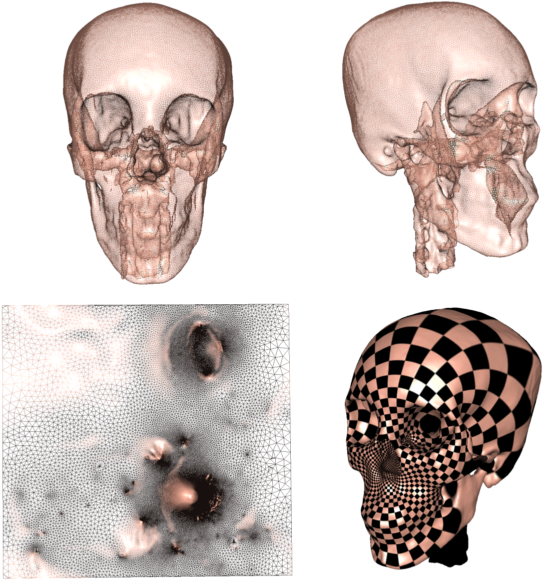
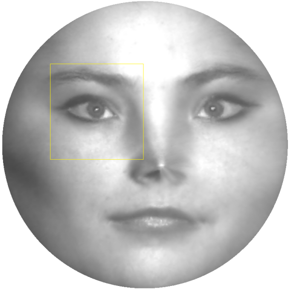
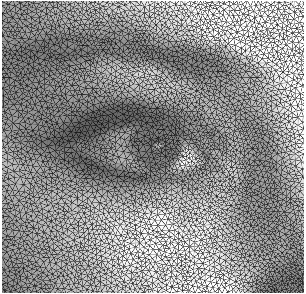
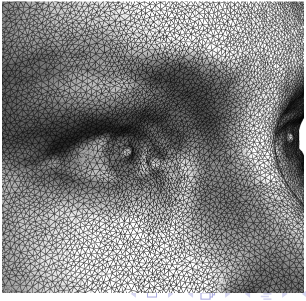


Figure: Conformal mapping by discrete surface Ricci flow.

Robustness Test - genus 39 anatomical model



Surface Remesh

Surface Remesh

Figure: Conformal mapping by discrete surface Ricci flow.

Surface Remesh

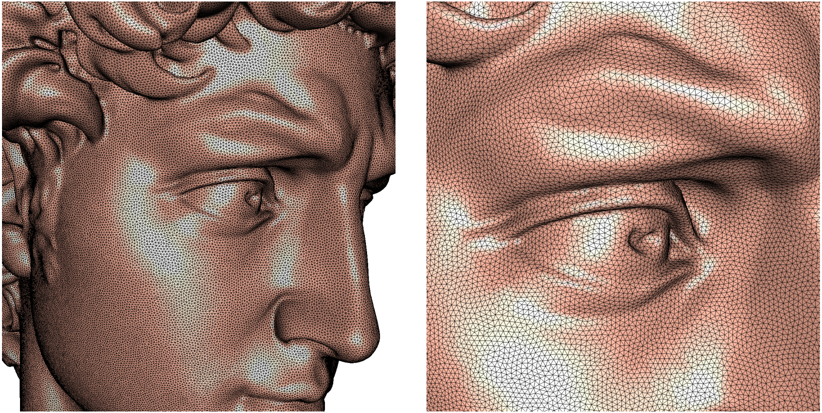


Figure: Conformal mapping by discrete surface Ricci flow.

Surface Remesh

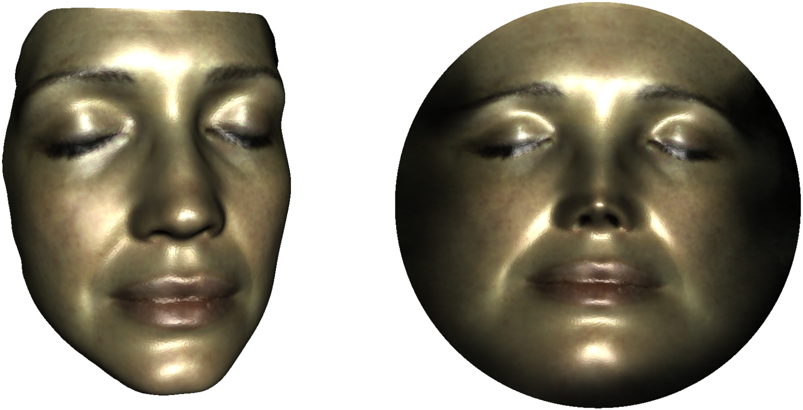


Figure: Conformal mapping by discrete surface Ricci flow.

Surface Remesh

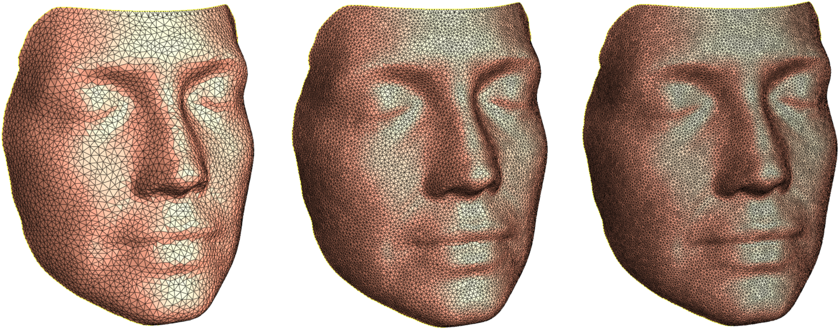
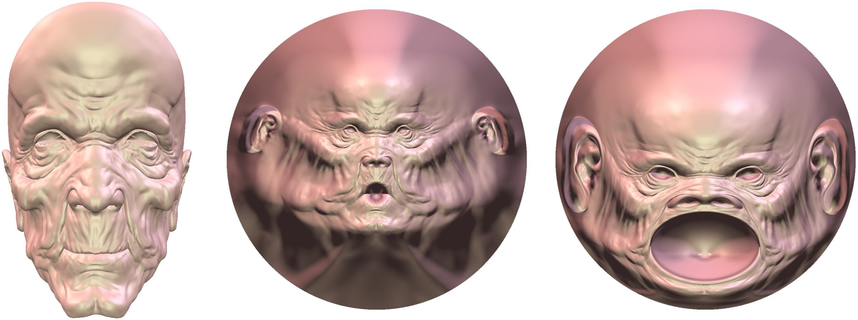


Figure: Multi-resolution Remeshing results.

Surface Multiresolution Compression



input surface conformal mapping OT mapping

Figure: Conformal and optimal transport mappings.

Surface Multiresolution Compression

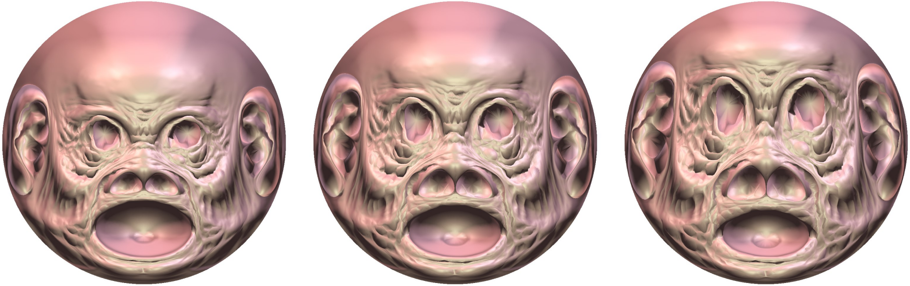


Figure: Optimal transport mappings, the target measures are weighted Gaussian curvature and the area element.

Surface Multiresolution Compression

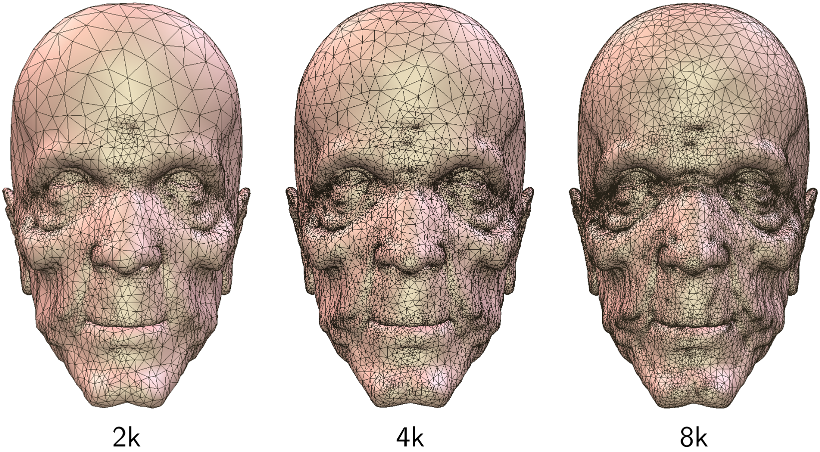


Figure: Multi-resolution Remeshing results.

Surface Multiresolution Compression

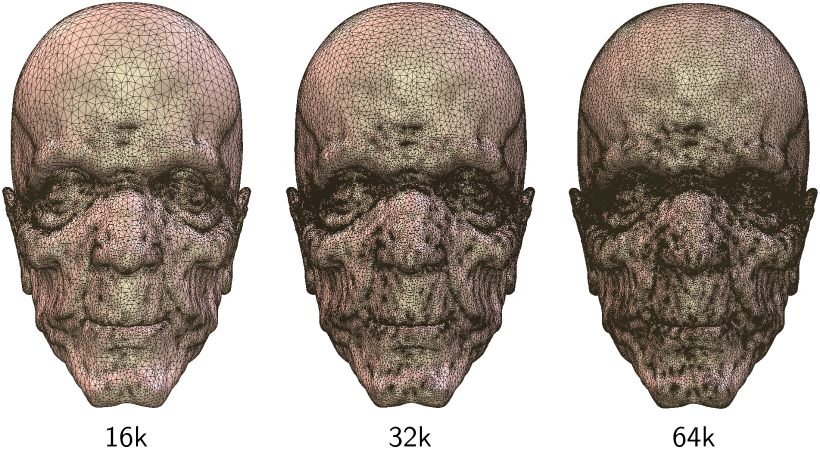


Figure: Multi-resolution Remeshing results.

Chassis Model

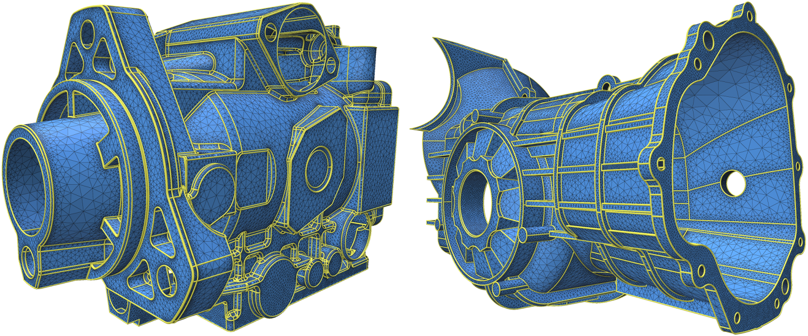
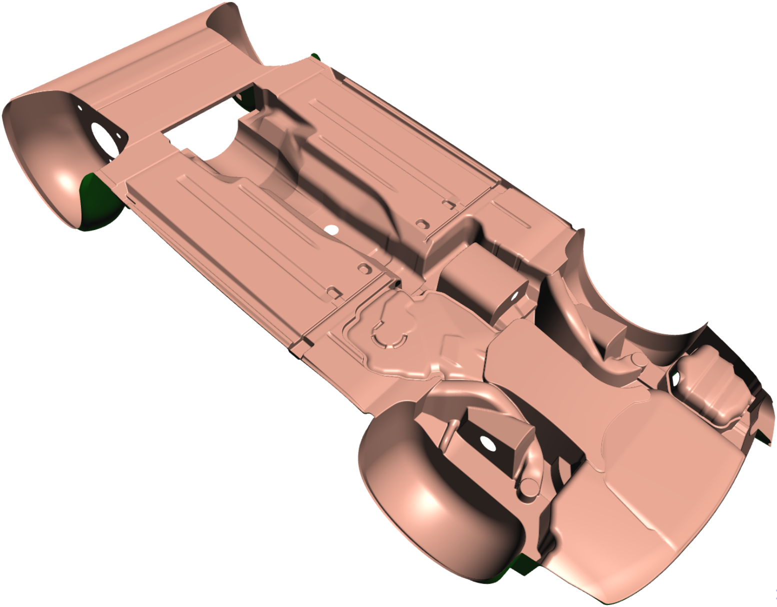
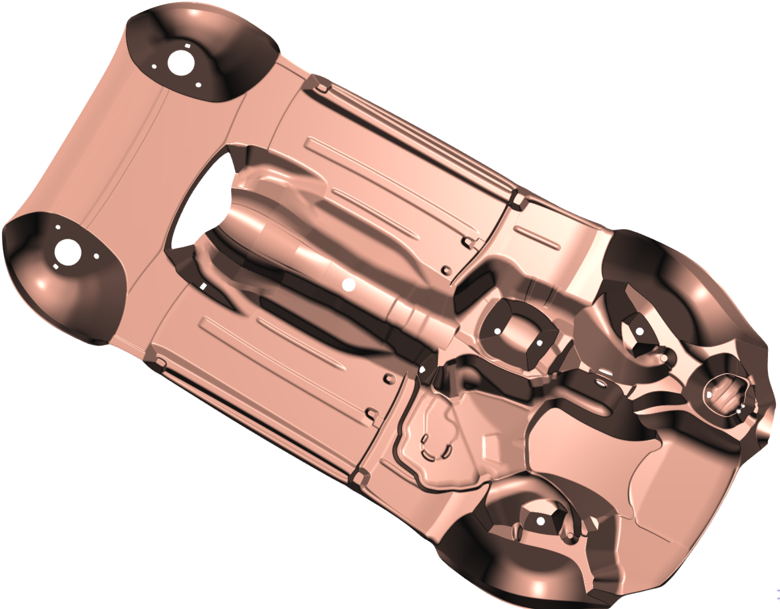


Figure: Remeshing a mechanical part.

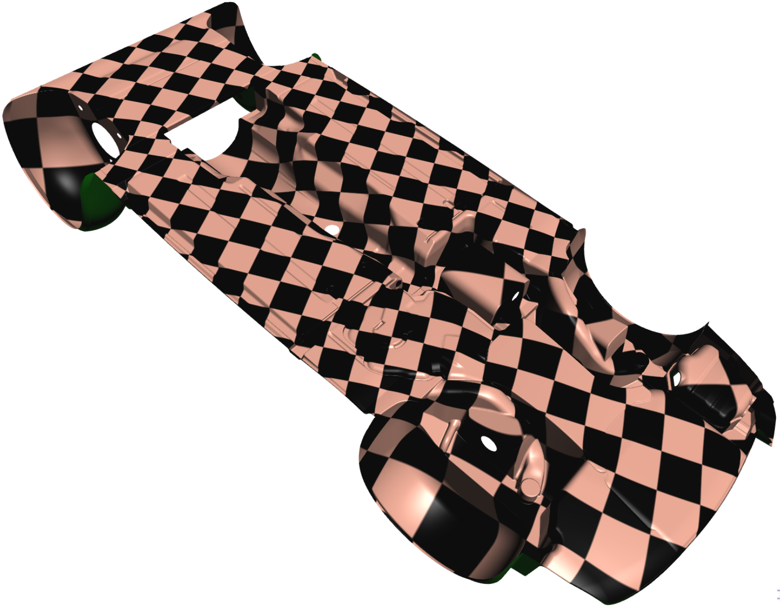
Chassis Model



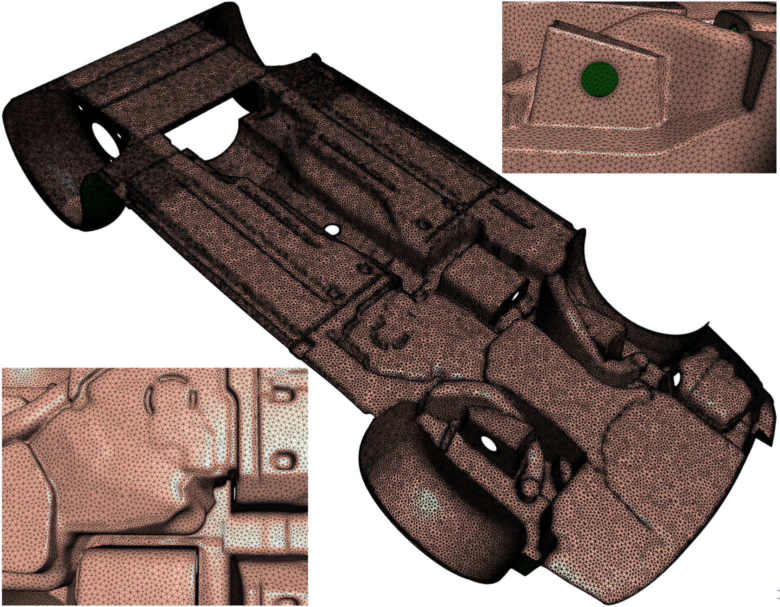
Chassis Model



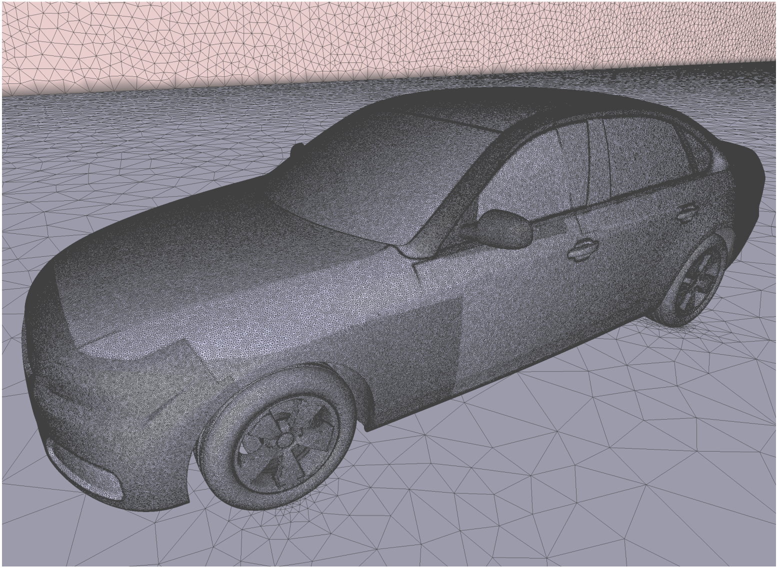
Chassis Model



Chassis Model

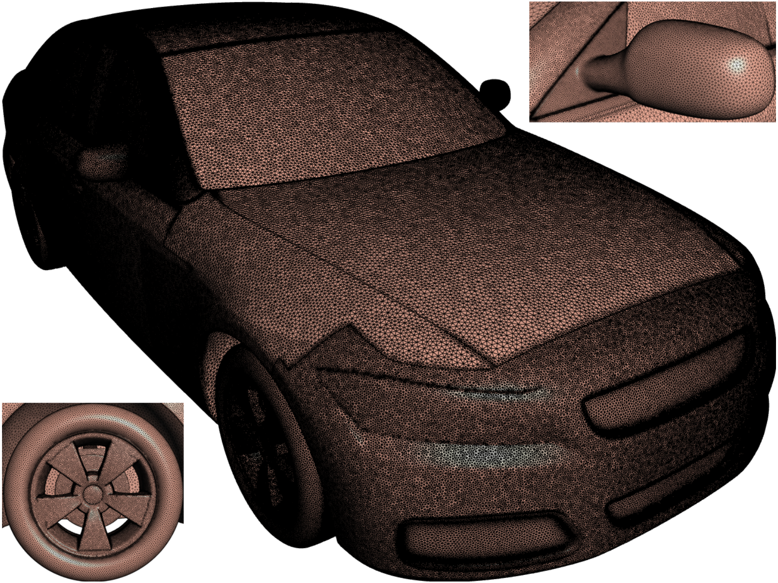


Car Model



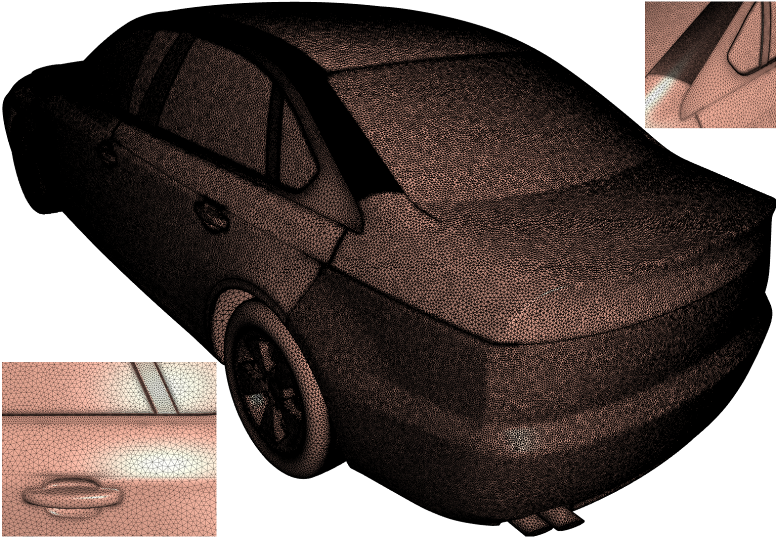
(d) remeshing result

Car Model



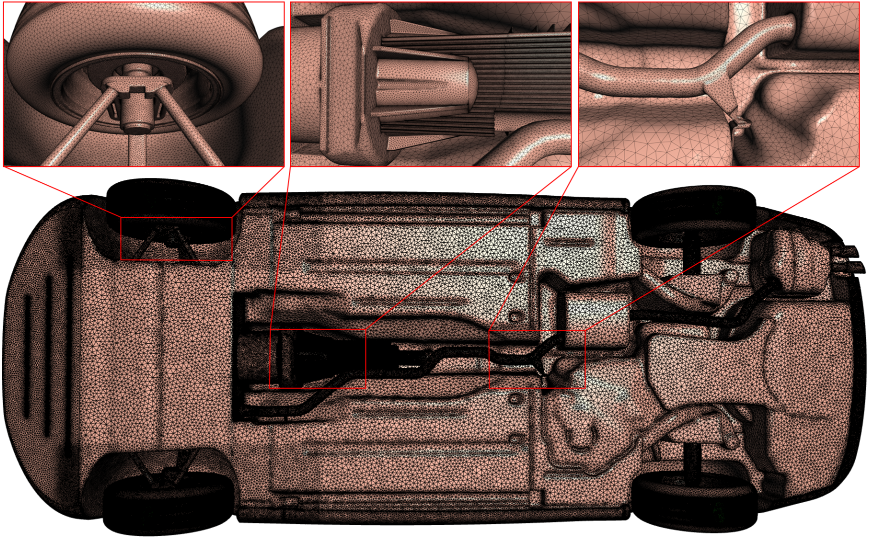
(d) remeshing result

Car Model



(d) remeshing result

Car Model



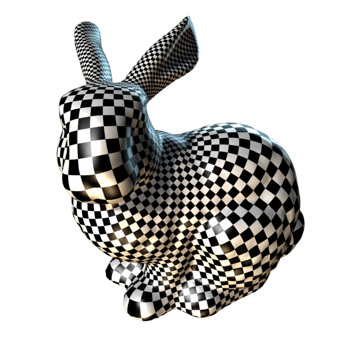
(d) remeshing result

References

- X. Gu, F. Luo, J. Sun and T. Wu, "A Discrete Uniformization Theorem for Polyhedral Surfaces", Journal of Differential Geometry, Volume 109, Number 2, Pages 223-256, 2018.
- Min Zhang, Ren Guo, Wei Zeng, Feng Luo, Shing-Tung Yau and Xianfeng Gu, **The Unified Discrete Surface Ricci Flow**, Graphical Models Volume 76, Issue 5, Pages 321-339, (2014).
- Xianfeng Gu, Ren Guo, Feng Luo, Jian Sun and Tianqi Wu, **A discrete uniformization theorem for polyhedral surfaces II**, Journal of Differential Geometry (JDG), Volume 109, Number 3, Pages 431-466, 2018.
- M. Ma, X. Yu, N. Lei, H. Si, X. Gu.**Guaranteed Quality Isotropic Surface Remeshing Based on Uniformization** IMR26, 18-21 September 2017, Barcelona, Spain

Thanks

For more information, please email to gu@cs.stonybrook.edu.



Thank you!