AN INITIAL REFINEMENT IS UNNECESSARY FOR A LINEAR CONFORMING CLOSURE OF THE REFINEMENT BY NEWEST VERTEX BISECTION

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ABSTRACT

An arbitrary n-dimensional simplicial complex is initialised for adaptive mesh refinement by standard bisection (aka *newest vertex bisection*) simply and quickly without an initial refinement splitting any original simplex into many ones. It requires an efficient algorithm that colours the nodes of the initial triangulation (any number of colours is admissible, fewer colours are better) as a missing link. The theorem of Binev, Dahmen, and DeVore about the ratio of the total number of bisected and the total number of marked simplices is extended to refinements of a triangulation initialised in that way.

Keywords: newest vertex bisection, standard bisection, adaptive mesh refinement, simplicial, BDV theorem, initialization problem

1. INTRODUCTION

Among various notations describing the standard bisection [1, 2, 3], we use Kossaczký's: An *embedding parallelotope* of a m-simplex $S = \operatorname{conv}(p_0, \dots, p_m)$ of *type* $k \in \{0, \dots, m\}$ is

$$P := p_0 + \sum_{j=1}^{k} [0,1](p_j - p_{j-1}) + [0,1](2p_{k+1} - p_0 - p_k) + \sum_{j=k+2}^{n} [0,2](p_j - p_{j-1}) \supset S,$$

i.e. there is a sequence of faces $F_k \subset \cdots \subset F_m = P$ of P with the properties:

- $\dim F_j = j$ for all $j = k, \dots, m$.
- The edges $\overline{p_0 p_1}, \dots, \overline{p_{k-1} p_k}$ are edges of F_k.
- For any $j = k{+}1, \dots, m$, the vertex p_j is the centre of F_j.

Note that P depends on the type and the order of the vertices. We encode this order and the embedding by the *T-array*

$$T = \begin{pmatrix} p_0 & \cdots & p_k \\ & p_{k+1} & \\ & \vdots & \\ & p_m & \end{pmatrix}.$$

Let a *diagonal* of a parallelotope P be a line segment between its vertices which runs through the relative interior of P. An embedded simplex S of type $k \geq 1$ is bisected at the edge $\overline{p_0 p_k}$, the diagonal of F_k, into two children S_1 and S_2. The children are type-$(k{-}1)$ embedded in the same parallelotope. The vertex $(p_0 + p_k)/2$ is the *new vertex* $N_{\text{new}}(S_1) = N_{\text{new}}(S_2)$ of the children. If S is type-0 embedded in the parallelotope $P = p_0 + \sum_{j=1}^{m} [0,2](p_j - p_{j-1})$, encoded by the T-array $(p_0 \ \cdots \ p_m)^T$, then S can be also type-m embedded into $p_0 + \sum_{j=1}^{m} [0,1](p_j - p_{j-1})$, i.e. P scaled at p_0 by factor $1/2$, which is encoded by $(p_0 \ \cdots \ p_m)$. With this scaled embedding, the bisection can continue. Any edge E between the vertices $p_0, \dots, p_k$ is a diagonal of a unique j-dimensional face (for some unique $j \in \{1, \dots, k\}$) of F_k. We say that E has *level 0* and *type*

j. Any other edge E of S is a diagonal of a unique j-dimensional face (for some unique $j \in \{1, \ldots, n\}$) of a 1/2-scaled copy of P which embeds some descendant of S. We say that E has *level 1* and *type* j.

Adaptive Finite Element Methods follow the loop:

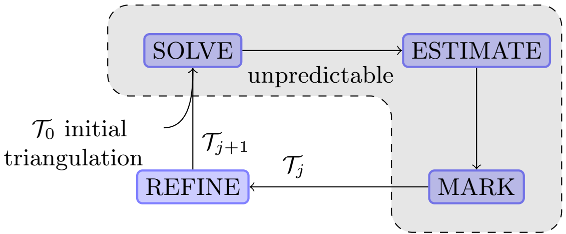

Figure 1: The AFEM loop

Start with some coarse conforming triangulation $\mathcal{T}_0$ of a polytope Ω with embeddings for each simplex $S \in \mathcal{T}_0$. Then repeat the following: SOLVE a discrete PDE on the coarse mesh $\mathcal{T}_j$ numerically. ESTIMATE the contribution of each simplex to the error. MARK a set of simplices $\mathcal{M}_j \subset \mathcal{T}_j$ with large estimates. REFINE the mesh bisecting (by standard bisection) at least the marked simplices $\mathcal{M}_j$ and as many further simplices as necessary such that the new mesh $\mathcal{T}_{j+1}$ is still conforming. Then start the next loop, until the error becomes as small as desired. For a detailed explanation, see [4]. The solution and estimation step are not discussed here, so the marking algorithm is supposed to be unpredictable.

A sequence $\mathcal{T}_0, \ldots, \mathcal{T}_N$ of triangulations together with a sequence $\mathcal{M}_0, \ldots, \mathcal{M}_{N-1}$ of marked simplices constructed in that way is called a *refinement sequence*. Under certain *initial conditions* for $\mathcal{T}_0$, [4] (for 2D) and [5] (for nD) showed the *BDV theorem* about the linear conforming closure: There exists a constant $C_{\mathrm{BDV}}(\mathcal{T}_0)$ such that for all refinement sequences,

$$\#\mathcal{T}_N - \#\mathcal{T}_0 \leq C_{\mathrm{BDV}} \sum_{j=0}^{N-1} \mathcal{M}_j.$$

The *initialisation* of an unstructured simplicial complex, i.e. equipping each simplex of a conforming triangulation with an embedding such that the initial conditions are satisfied, has been an often discussed problem since the 90s. For dimension 2, [6] have given an efficient initialisation algorithm. For dimension n, [1, 5] suggested *initial refinements* as follows: At the beginning, every simplex is divided into $(n+1)!/2$ simplices, which are equipped with embeddings. Unfortunately, this leads to very acute angles. To avoid an initial refinement, [7] gave an initialisation which satisfies a weaker initial condition. The output of their algorithm has conforming uniform n-fold refinements by standard bisection, but a BDV theorem for it is unknown. For dimension 3, [8] used a slight extension of the standard bisection, equipped an arbitrary mesh with his extended embeddings, violating the initial conditions, and proved the BDV theorem for its refinements. Schön also gave an example of a triangulation which cannot be initialised satisfying the initial conditions.

This work presents a simple new initialisation algorithm for standard bisection of an arbitrary simplicial complex without initial refinements. The algorithm presupposes a colouring of the vertices of the initial triangulation with any number c of colours and given such a colouring, it runs in $O(c\#\mathcal{T}_0)$ time. It satisfies the strong initial condition in a broader sense and it satisfies the BDV theorem.

2. THE COLOUR INITIALISATION

The initialisation algorithm calls a function FIND_COLOURING to find a certain colouring $\mathfrak{c}$ as described below. A (good) algorithm for that is still missing. Let $\mathcal{N}(S)$ and $\mathcal{N}(\mathcal{T})$ denote the vertices of a simplex S and a triangulation $\mathcal{T}$, respectively. We initialise the mesh according to the *colour initialisation* (Algorithm 1):

Remark 1. If we find a colouring with only n colours, the algorithm does not add any virtual vertices and the initialised mesh satisfies the initial conditions given in [1] and [5]. The other extreme would be to use as many colours as vertices, which turns out as a special case of the algorithm in [7].

Definition 2. The output of this algorithm is the *initial virtual embedded triangulation* $\mathcal{V}_0$. It is refined by the above loop. The set of all virtual embedded simplices generated from $\mathcal{V}_0$ by standard bisection is the *set of admissible virtual embedded simplices* $\mathbb{V}$. For $V \in \mathbb{V}$, the *generation* $g(V)$ counts how often an initial simplex has to be bisected to become V. If an n-dimensional face of a virtual simplex V lies in Ω, it is called its *real face* $R(V)$. A real mesh, a conforming refinement of $\mathcal{T}_0$, appears as the set of real faces of a conforming refinement of $\mathcal{V}_0$. If a real face is marked, the appendant virtual simplex has to be bisected successively, until the real face is bisected.

Remark 3. An implementation can completely dispense with the virtual vertices and simplices. They serve only theoretical purposes here.

3. INITIAL CONDITIONS

Definition 4. According to [1, (A1) and (A2) on p. 282] and with a similar effect as [5, (a) and (b) on p. 232], the *initial conditions (IC)* are:

Algorithm 1 Colour initialisation

function FIND_COLOURING(Conforming triangulation $\mathcal{T}$)

return colouring $\mathfrak{c} : \mathcal{N}(\mathcal{T}) \to \{1, \dots, c\}$ (for an arbitrary $c \in \{n, n+1, \dots\}$), such that in each simplex S:

- there are at most 2 vertices $p \in \mathcal{N}(S)$ with $\mathfrak{c}(p) = 1$,
- for any other colour 2, …, c, there is at most one vertex $p \in \mathcal{N}(S)$ with this colour.

end function

function INITIALISE(Initial conforming triangulation $\mathcal{T}_0$)

Let $\mathfrak{c} :=$ FIND_COLOURING($\mathcal{T}_0$).

Embed Ω into a high dimensional space (e.g. into $\mathbb{R}^c$) isometrically. Add *virtual vertices* to each simplex and extend the colouring $\mathfrak{c}$ to them until each simplex contains two vertices of colour 1 and one vertex of the other colours each. The result is called a *virtual simplex*. Any virtual vertex appears in only one simplex. Place them in a way that the intersection of any two simplices is not enlarged through the addition of the virtual vertices.

In each virtual simplex, let p_0 and p_1 be the two vertices of color 1 in arbitrary order. Then every virtual simplex is embedded according to the T-array

$$\begin{pmatrix} p_0 \quad p_1 \\ \mathfrak{c}^{-1}(2) \\ \vdots \\ \mathfrak{c}^{-1}(c) \end{pmatrix}.$$

return embedded virtual simplices

end function

1. $\mathcal{V}_0$ is conforming.
2. All embeddings in $\mathcal{V}_0$ are of the same type.
3. If any edge E occurs in two initial embedded virtual simplices $U, V \in \mathcal{V}_0$, then E has the same level and type in U as it has in V.

Remark 5. The output of the colour initialisation satisfies the IC obviously.

4. THE BDV THEOREM

The BDV theorem can be proven adapting the proof of Binev, Dahmen, and DeVore by means of the following stepping stones:

Proposition 6. *1. If a virtual embedded simplex is bisected* $\max\{2c-2n, c-n+1\}$ *times successively, any real face of it is at least once bisected.*

2. Let $|\cdot|$ denote the n-dimensional Lebesgue measure. There are only finitely many values for the scaled volume $2^{\frac{n}{c}g(V)}|R(V)|$ and diameter $2^{g(V)/c}$ diam $R(V)$ *among the admissible virtual simplices $\mathbb{V}$, especially a minimum d for the first value and a maximum D for the second value.*

3. If the marking of the embedded simplex $V \in \mathbb{V}$ causes the bisection of another $U \in \mathbb{V}$ in the refinement step, then for any child V' of V and any child U' of U, there is a sequence $V' = V_0, \dots, V_N = U'$ of virtual simplices in $\mathbb{V}$ with the properties:

$$\begin{aligned} g(V_1) &= g(V_0), \\ g(V_j) &= g(V_{j-1}) - 1 \quad \textit{for } j = 2, \dots, N, \\ R(V_{j-1}) \cap R(V_j) &\neq \emptyset \qquad \textit{for } j = 1, \dots, N. \end{aligned}$$

4. If the marking of V causes the bisection of U, then for a child V' of V it holds that

$$U \subset B\left(N_{\text{new}}(V'), D \sum_{j=g(U)+1}^{g(V)+1} 2^{-j/c}\right).$$

Theorem 7. *For any refinement sequence $\mathcal{T}_0$, …, $\mathcal{T}_N$ with marked simplices $\mathcal{M}_0$, …, $\mathcal{M}_{N-1}$ of the real mesh, it holds that*

$$\#\mathcal{T}_N - \#\mathcal{T}_0 \le C_{BDV}(n, c, d, D) \sum_{j=1}^{N} \#\mathcal{M}_j.$$

References

[1] Kossaczký I. "A recursive approach to local mesh refinement in two and three dimensions." *Journal of Computational and Applied Mathematics*, vol. 55, no. 3, 275 – 288, 1994. URL http://www.sciencedirect.com/science/article/pii/0377042794900345

[2] Maubach J.M.L. "Local bisection refinement for n-simplicial grids generated by reflection." *SIAM Journal on Scientific Computing*, vol. 16(1), 210–227, 1995

[3] Traxler C.T. "An algorithm for adaptive mesh refinement in n dimensions." *Computing*, vol. 59, 115–137, 1997

[4] Binev P., Dahmen W., DeVore R. "Adaptive finite element methods with convergence rates." *Numerische Mathematik*, vol. 97(2), 219–268, 2004

[5] Stevenson R. "The Completition of Locally Refined Simplicial Partitions Created by Bisection." *Mathematics of Computation*, vol. 77, 227–241, 2008

[6] Biedl T.C., Bose P., Demaine E.D., Lubiw A. "Efficient algorithms for Petersen's matching theorem." *J. Algorithms*, vol. 38, no. 1, 110–134, 2001

[7] Alkämper M., Gaspoz F., Klöfkorn R. "A Weak Compatibility Condition for Newest Vertex Bisection in Any Dimension." *SIAM Journal on Scientific Computing*, vol. 40, no. 6, A3853–A3872, 2018. URL `https://doi.org/10.1137/17M1156137`

[8] Schön P. *Scalable adaptive bisection algorithms on decomposed simplicial partitions for efficient discretizations of nonlinear partial differential equations*. Ph.D. thesis, Univ. Freiburg, Fakultät für Mathematik und Physik, 2017. URL `d-nb.info/1159877963/34`