SURFACE CROSS FIELD GENERATION BASED ON CHARACTERISTIC CLASS OF FIBER BUNDLE

Zhongxuan Luo[1] Yiming Zhu[1] Xiaopeng Zheng[1]* Na Lei[1] Hang Si[2] Xianfeng Gu[3]

[1] *Dalian University of Technology, Dalian, Liaoning, China*
[2] *Cadence Design Systems, Austin TX, U.S.A.*
[3] *SUNY at Stony Brook, Stony Brook, NY, U.S.A.*

ABSTRACT

Constructing smooth cross fields on surfaces plays a crucial role for structured quad-mesh generation. This work develop a theoretical framework, which treats a cross field as a section of the cross bundle over the surface, the singularities as the topological obstruction to the existence of global section, namely the characteristic class of the bundle. The work proves the necessary and sufficient condition for the singularity configuration of cross fields based on Ricci flow, which leads to a rigorous and efficient algorithm for cross field construction. Experimental results demonstrate the efficiency and efficacy of the algorithm.

Keywords: surface cross field generation, fiber bundle, characteristic class, topological obstruction, holonomy, harmonic forms, Ricci flow

1. INTRODUCTION

Surface cross field construction is a fundamental problem in geometric modeling, which is often applied for quad-mesh generation in CAD/CAE fields.

There are intensive researches concentrated on cross fields based on their different representations, such as spherical harmonic basis[1], N-Rosy[2, 1], complex function[3] and seamless parameterization [4]. The singularity configurations have been studied using Poincare-Hopf theorem [5] and Abel-Jacobi theorem [6, 7]. Our current work generalized the Hopf-Poincaré theorem to the cross field.

A cross field is a generalization of an orthonormal frame field on the surface. From the point of view of modern algebraic topology [8], a frame field can be treated as a section of the orthnormal frame bundle of the surface. In general, there is no global section of the frame bundle, the obstruction is represented by the singularities of the frame field, which can be treated as a cohomological class in the cohomology group $H^2(S,\mathbb{Z})$, the so-called characteristic class of the othronormal frame bundle. Different frame fields are different sections of the same fiber bundle, and have different singularity configurations. But if the singularity configurations are treated as cohomological classes, they are identical. The surface orthonormal frame bundle is a special circle bundle of the surface, non-isomorphic circle bundles have different characteristic classes.

In this work, we systematically generalize the theoretic framework to cross fields, each cross field is treated as a section of the cross bundle over the surface, the singularity configuration is the topological obstruction to the existence of global section [8]. Different cross fields give different singularity configurations, they are identical in the cohomology group $H^2(S,\mathbb{Z})$, namely the characteristic class of the cross bundle. Furthermore, we give the sufficient and necessary conditions for the singularity configuration which leads to a rigor-

*Corresponding author: zhengxp@dlut.edu.cn

ous and efficient algorithm for constructing cross fields with prescribed singularities.

The algorithm computes a flat Riemannian metric using Ricci flow [9, 10], such that all the curvatures are concentrated on the singularities. A cross at the base point can be parallel transported everywhere on the surface. A special harmonic 1-form is constructed to compensate the holonomy.

The algorithm has been validated by constructing smooth cross fields on surfaces with various topologies. The experimental results show the efficiency and efficacy of the method.

2. THEORETIC FOUNDATION

This section briefly introduces the theoretic background for the characteristic class theory for fiber bundles. For more details, we refer readers to [8] for more thorough treatments.

2.1 Fiber Bundle and Characteristic Class

Fiber Bundle The fiber bundle and its section play fundamental roles.

Definition 2.1 (Group Action)**.** A topological group G acts on a space X if there is a group homeomorphism $G \to \mathrm{Homeo}(X)$ such that the adjoint

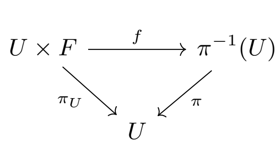
$$G \times X \to X \quad (g, x) \mapsto g(x)$$

is continuous. We usually write $g \cdot x$ instead of $g(x)$.

An action is called *free* if $g(x) \neq x$ for all $x \in X$ and for all $g \neq e$. An action is *effective* if the homomorphism $G \to \mathrm{Homeo}(X)$ is injective.

Definition 2.2 (Fiber Bundle)**.** Let G be a topological group acting effectively on a space F. A *fiber bundle E over B with fiber F and structure group G* is a map $\pi : E \to B$ together with a collection of homeomorphisms $\{\varphi : U \times F \to \pi^{-1}(U)\}$ for open sets U in B, φ is called a *a chart over* U, such that

1. The diagram



   commutes for each chart φ over U.

2. Each point of B has a neighborhood over which there is a chart.

3. If φ is a chart over U and $V \subset U$ is open, then the restriction of φ to V is a chart over V.

4. For any charts φ, φ' over U, there is a continuous map $\theta_{\varphi,\varphi'} : U \to G$ so that
$$\varphi'(u, f) = \varphi(u, \theta_{\varphi,\varphi'}(u) \cdot f), \forall u \in U, \forall f \in F.$$
The map $\theta_{\varphi,\varphi'}$ is called the *transition function* for φ, φ'.

5. The collection of charts is maximal among collections satisfying the previous conditions.

A *section* of the fiber bundle $\pi : E \to B$ is a continuous map, $\sigma : B \to E$, such that $\pi \circ \sigma = id$.

$$\pi(\sigma(x)) = x \quad \forall x \in B.$$

Cross Bundle and Cross Field Suppose M is a Riemannian manifold, namely each tangent space T_pM is assigned with an inner product. Let $\mathcal{F}^o_p(M)$ be the space of all possible orthonormal frames of T_pM, then

$$\mathcal{F}^o(M) = \bigcup_{p \in M} \mathcal{F}^o_p(M) \xrightarrow{\pi} M$$

is called the orthonormal frame bundle of M. The structure group is $SO(n)$.

Suppose the base manifold is a surface $(S, \mathbf{g})$, on each tangent plane T_pS, all the orthonormal frames form the space $\mathcal{F}^o_p(S)$. A subgroup N of $SO(2)$ is defined as

$$N = \left\{ e^{i\frac{k\pi}{2}}, k = 0, 1, 2, 3 \right\}.$$

N acts on $\mathcal{F}^o_p(S)$, each orbit is called a *cross*, the quotient space is called the cross space,

$$\mathcal{C}^o_p(S) := \mathcal{F}^o_p(S)/N.$$

The *cross bundle* of the surface is defined as

$$\mathcal{C}^o(S) = \bigcup_{p \in S} \mathcal{C}^o_p(S) \xrightarrow{\pi} S,$$

namely $\mathcal{C}^o(S) = \mathcal{F}^0(S)/N$. A section $\sigma : S \to \mathcal{C}^o(S)$ is called a *cross field* on S.

Topological Obstruction Given a fiber bundle $\pi : E \to B$, there may not exist any global section $\sigma : B \to E$, the obstruction can be represented as the characteristic class of the bundle. For convenience, we assume the base space B is a CW-complex, the fiber space F is path connected.

Definition 2.3 (CW-complex)**.** A CW complex is constructed by taking the union of a sequence of topological spaces

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots X_n = X,$$

such that each X_k is obtained from X_{k-1} by gluing copies of k-cells e^k_α, each homeomorphic to D^k, to X_{k-1} by continuous gluing maps $g^k_\alpha : \partial e^k_\alpha \to X_{k-1}$. Each X_k is called the k-skeleton of the complex.

The global section $\sigma : B \to E$ can be constructed step by step:

1. At the initial step, we define the section on the 0-skeleton X_0, namely for each point $e^0_\alpha \in X_0$, we choose a point $\sigma(e^0_\alpha)$ in the fiber F at e^0_α;

2. At the 1-st step, we extend σ to X_1. For each cell $e^1_\alpha \in X_1$, we find a path γ in $\pi^{-1}(e^1_\alpha) \subset E$, γ connects $\sigma(\partial e^1_\alpha)$ defined at the initial step;

3. At the 2-nd step, we extend σ to X_2. For each cell $e^2_\alpha \in X_2$, $\sigma(\partial e^2_\alpha)$ has been defined in the 1st step, which is a loop in the fiber space F. If $\sigma(\partial e^2_\alpha) = e$, then σ can be extended into the interior of e^2_α, otherwise, we encounter an obstruction. Therefore we define a $\pi_1(F)$-valued 2-form $\theta^2(\sigma)$,

$$\theta^2(\sigma)(e^2_\alpha) = [\sigma(\partial e^2_\alpha)] \in \pi_1(F).$$

4. At the k-th step, we extend σ to X_k. For each cell $e^k_\alpha \in X_k$, $\sigma(\partial e^k_\alpha)$ has been defined in the $k-1$-th step, we obtain a $\pi_{k-1}(F)$-valued k-form $\theta^k(\sigma)$,

$$\theta^k(\sigma)(e^k_\alpha) = [\sigma(\partial e^k_\alpha)] \in \pi_{k-1}(F).$$

   $\theta^k(\sigma)$ is the topological obstruction class, which determines whether σ can be extended to X_k.

Theorem 2.1 (Topological Obstruction). *If $\pi_{k-1}(F)$ is trivial, then σ can be extended to X_k; otherwise, if $\pi_{k-1}(F)$ is Abelian, the obstruction class $\theta^k(\Sigma)$ is 0, then we can modify σ on X_{k-1} then σ can be extended to X_k. Otherwise, σ can not be extended.*
Furthermore, if σ_1 and σ_2 are two different constructions of global sections, then there is a $\pi_{k-1}(F)$ valued $(k-1)$-form on B, such that

$$\theta^k(\sigma_1) - \theta^k(\sigma_2) = d\omega.$$

Namely, the $\theta^k(\sigma_1)$ and $\theta^k(\sigma_2)$ are cohomological equivalent.

Therefore, we call the class $[\theta^k(\sigma)] \in H^k(B, \pi_{k-1}(F))$ the topological obstruction class of the bundle $\pi : E \to B$, which is also the characteristic class of the bundle.

Characteristic Class of Surface Cross Bundle We apply the above topological obstruction theory to study the cross fields on a closed metric surface $(S, \mathbf{g})$. The cross space at each point $\mathcal{C}^o_p(S)$ is a topological circle $\mathbb{S}^1$,

$$\mathcal{C}^o_p(S) = \mathcal{F}^o_p(S)/N = \left\{ e^{i\tau} : \tau \in [0, \frac{\pi}{2}) \right\}.$$

Therefore its fundamental group $\pi_1(\mathcal{C}^o_p(S)) = \mathbb{Z}$. Each cross field is a global section σ of the cross bundle $\mathcal{C}^o(S)$, the singularities of σ with indices form a 0-form, $\theta^0(\sigma) = \sum_i \lambda_i p_i$, where $\lambda_i \in \pi_1(\mathcal{C}^o_p(S)) = \mathbb{Z}$. 0-forms and 2-forms are dual to each other, therefore $\theta(\sigma)$ can be treated as a 2-form in $H^2(S, \pi_1(\mathcal{C}^o_p(S))) = H^2(S, \mathbb{Z})$,

$$\theta^2(\sigma) = \sum_i \lambda_i \Delta_i, \quad \lambda_i \in \mathbb{Z},$$

where each singular point p_i is inside the triangular face Δ_i, $p_i \in \Delta_i$, the cross field σ maps $\partial\Delta_i$ to the fiber $\mathcal{C}^0_{p_i}$, $\sigma_\# : \pi_1(\partial\Delta_i) \to \pi_1(\mathcal{C}^0_{p_i})$, $\mathbb{Z} \to \mathbb{Z}$, $z \mapsto \lambda_i z$. Namely, the index λ_i of p_i is the winding number of σ at the singularity. Hence, we obtain the following sufficient and necessary condition for the cross field singularities:

Theorem 2.2 (Cross Field Singularity). *Suppose $(S, \mathbf{g})$ is an orientable, closed metric surface. Given a 0-form $\theta = \sum_{i=1}^n \lambda_i p_i$, then θ is the singularity configuration of a continuous cross field on S, if and only if*

$$\sum_{i=1}^n \lambda_i = 4\chi(S), \tag{1}$$

where $\chi(S)$ is the Euler characteristic number.

Proof. Necessary Condition: suppose σ is a global continuous cross field on S, and its obstruction class is $\theta^2(\sigma)$. We can find a tangential vector field τ', τ' can be treated as a special case of cross field, but each index differs by a factor 4. Then $[\theta^2(\sigma)] = [\theta^2(\sigma')]$. By Hopf-Poincaré vector field index theorem, we know the total index of σ' is $\chi(S)$ treated as a vector field, and $4\chi(S)$ treated as a cross field. Hence the total index of σ is $4\chi(S)$, Eqn. (1) holds.

Sufficient Condition Given $\theta = \sum_i \lambda_i p_i$, we set target discrete Gaussian curvature $\bar{K}(p_i) = \lambda_i \pi/2$ and zero at the other points. Then the total target curvature satisfies the Gauss-Bonnet condition $\sum_i \bar{K}(p_i) = 2\pi\chi(S)$. According to the discrete Ricci flow theorem [9], there is a flat metric $\bar{\mathbf{g}}$ with cone singularities at p_i's, conformal to the original metric. We choose a base point $q \in S \setminus \{p_i\}$. Suppose the fundamental group generators of S are $\gamma_1, \gamma_2, \cdots, \gamma_{2g-1}, \gamma_{2g}$, where g is the genus of the surface. We fix a cross $c \in T_q S$, and parallel transport the cross c along γ_k, when the cross c returns to the based point q, the rotation angle is β_k, we say the *cross holonomy* along γ_k is β_k. We compute the basis of the deRham cohomology group $H^1_{dR}(S, \mathbb{R})$, $\omega_1, \omega_2, \cdots, \omega_{2g}$, which are harmonic and satisfy

$$\int_{\gamma_i} \omega_j = \delta^j_i, \quad \forall 1 \le i, j \le 2g.$$

Then we define a harmonic 1-form

$$\omega = \beta_1\omega_1 + \beta_2\omega_2 + \cdots + \beta_{2g-1}\omega_{2g-1} + \beta_{2g}\omega_{2g}.$$

Then we construct the global smooth cross field σ as follows: fix a cross c at the base point q, for any point p on the punctured surface $S \setminus \{p_i\}$, find a path $\gamma \subset S \setminus \{p_i\}$ from q to p, we parallel transport c along γ to p to get c', then rotate c' clockwisely by an angle $\int_\gamma \omega$ to obtain $\sigma(p)$. This procedure produces a global smooth cross field σ, with singularity configuration $\theta^2(\sigma)$, which is dual to the given θ. □

3. COMPUTATIONAL ALGORITHMS

This section briefly introduces the computational algorithms, the computational topological algorithms, such as fundamental group generator, harmonic cohomology group basis can be found in [11, 12]. The algorithmic details for Ricci flow can be found in [9, 10].

Algorithm 1 Cross Field Construction

Require: Closed Triangle mesh M, singularities $\theta = \sum_i \lambda_i p_i$

Ensure: Cross field σ with prescribed singularities θ

1. Set target curvature $\bar{K}_i = \lambda_i \pi/2$
2. Compute a flat metric $\bar{\mathbf{g}}$ with target curvature
3. Choose a base point $q \in S \setminus \{p_i\}$, compute the generators of fundamental group $\{\gamma_1, \gamma_2, \ldots, \gamma_{2g}\}$
4. Parallel transport a fixed cross c at the base point q along γ_i's to compute the holonomy β_k
5. Compute harmonic 1-form basis of $H^1_{dR}(M, \mathbb{R})$ $\{\omega_1, \omega_2, \ldots, \omega_{2g-1}, \omega_{2g}\}$, such that $\int_{\gamma_i} \omega_j = \delta_i^j$
6. Construct a harmonic 1-form $\omega = \sum_i \beta_i \omega_i$

for each vertex $v_i \in M$ **do**
  a. Find a path $\gamma \subset S \setminus \{q_i\}$ from q to v_i
  b. Parallel transport c along γ to obtain c'
  c. Rotate c' by angle $\int_\gamma \omega$ clockwisely
end for

4. EXPERIMENTAL RESULTS

We have tested the proposed algorithm on various 3D models with complicated topologies as shown in Fig. 1. The singularities are marked with different colors, the red, blue and green circles represent the indices of $+1$, -1 and -2 respectively. It is easy to see that the cross field is globally smooth. The second row first frame shows a cross field on a torus with two singularities, one is of index of $+1$ and the other is of index -1 as shown in Fig. 1. The algorithm is efficient enough. For the genus 3 **Buddha** model with 118.7k faces, the cross field construction only takes 42.351 seconds.

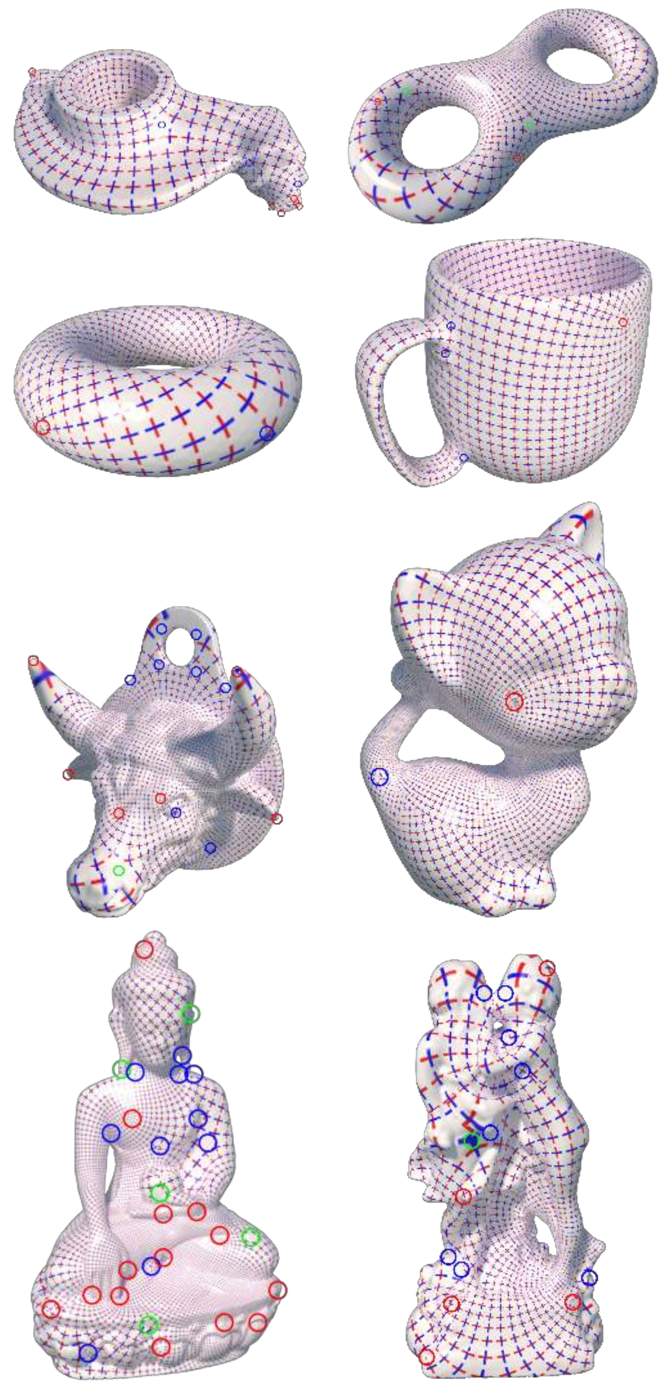


Figure 1: Cross fields construction on surfaces with complicated topologies.

References

[1] Zhang P., Vekhter J., Chien E., Bommes D., Vouga E., Solomon J. "Octahedral Frames for Feature-Aligned Cross Fields." *ACM Transactions on Graphics (TOG)*, vol. 39, no. 3, 1–13, 2020

[2] Palacios J., Zhang E. "Rotational symmetry field design on surfaces." *ACM Transactions on Graphics (TOG)*, vol. 26, no. 3, 55–es, 2007

[3] Kowalski N., Ledoux F., Frey P. "A PDE based approach to multidomain partitioning and quadrilateral meshing." *Proceedings of the 21st international meshing roundtable*, pp. 137–154. Springer, 2013

[4] Shen H., Zhu L., Capouellez R., Panozzo D., Campen M., Zorin D. "Which cross fields can be quadrangulated? Global parameterization from prescribed holonomy signatures." *ACM Transactions on Graphics (TOG)*, vol. 41, no. 4, 1–12, 2022

[5] Fogg H.J., Sun L., Makem J.E., Armstrong C.G., Robinson T.T. "Singularities in structured meshes and cross-fields." *Computer-Aided Design*, vol. 105, 11–25, 2018

[6] Lei N., Zheng X., Luo Z., Luo F., Gu X. "Quadrilateral mesh generation II: Meromorphic quartic differentials and Abel–Jacobi condition." *Computer methods in applied mechanics and engineering*, vol. 366, 112980, 2020

[7] Zheng X., Zhu Y., Chen W., Lei N., Luo Z., Gu X. "Quadrilateral mesh generation III: Optimizing singularity configuration based on Abel–Jacobi theory." *Computer Methods in Applied Mechanics and Engineering*, vol. 387, 114146, 2021

[8] Davis J.F., Kirk P. *Lecture Notes in Algebraic Topology.* Graduate Studies in Mathematics. 2001

[9] Gu X., Luo F., Sun J., Wu T. "A Discrete Uniformization Theorem for Polyhedral Surfaces I." *Journal of Differential Geometry (JDG)*, vol. 109, no. 2, 223–256, 2018

[10] Gu X., Guo R., Luo F., Sun J., Wu T. "A Discrete Uniformization Theorem for Polyhedral Surfaces I." *Journal of Differential Geometry (JDG)*, vol. 109, no. 3, 431–466, 2018

[11] Gu X., Yau S.T. *Computational Conformal Geometry*, vol. 3 of *Advanced Lectures in Mathematics.* International Press and Higher Education Press, 2007

[12] Gu X., Yau S.T. *Computational Conformal Geometry.* International Press and Higher Education Press, 2020