SURFACE REMESHING BASED ON CONFORMAL UNIFORMIZATION

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ABSTRACT

Surface meshing plays a fundamental role in CAD and CAE. This work introduces a rigorous and efficient algorithm for generating high quality triangulations on surfaces with complicated topologies based on conformal geometry. The key idea is to flatten the input surface onto a planar domain by an angle-preserving (conformal) map, then a high quality planar mesh is generated and pulled back to the 3D surface. Since the mapping is angle-preserving, the Delaunay property is preserved to the surface triangulation. Furthermore, the planar sampling density is adaptive to the surface area element and the curvature, this guarantee the surface meshing quality. By converting the 3D meshing problem to the 2D problem, this method reduces the algorithmic difficulty and improves the computational efficiency and mesh quality. Our experimental results demonstrate the efficiency and efficacy of the proposed algorithm.

Keywords: surface mesh generation, conformal mapping, uniformization, Ricci flow, Delaunay refinement, adaptive sampling

1. INTRODUCTION

Surface meshing plays a crucial role in CAD/CAE fields. Although there are many mature planar mesh generation algorithms, such as the methods based on Delaunay refinement [1], centroidal voronoi diagram [2], surface mesh generation remains challenging due to the topological and geometric complexities.

Given a smooth surface $(S, \mathbf{g})$ embedded in the Euclidean space $\mathbb{R}^3$, a surface meshing algorithm should generate a sequence of triangulated polyhedral surfaces M_k, such that the discrete surface sequence $\{M_k\}$ converges to the smooth surface S under different metrics. The most commonly used convergence is measured by the Gromov-Hausdorff distance, but this convergence doesn't guarantee the convergences of the surface area, the geodesic lengths, the Laplace-Beltrami spectrum and so on. A more rigorous convergence is measured by the normal cycle convergence [3, 4], this guarantees the convergences of the surface area, geodesics, Gaussian curvature, principle curvatures and the Laplace-Beltrami spectrum and the eigen functions. The sufficient condition for normal-cycle convergence is that: the surface triangulations are geodesic Delaunay and the geodesic circum-radii uniformly converge to zero [4, 5]. Therefore it is crucial to generate geodesic Delaunay triangulations with the uniform sampling density not only on the surface, but also on the normal cycle of the surface.

In this work, we propose a novel and rigorous method to tackle this fundamental challenge. The key idea is to convert the surface meshing problem to the planar meshing problem via conformal mapping. According to the surface uniformization theorem [6, 7], all surfaces in reality can be conformally flattended to planar domains globally, therefore we can compute planar Delaunay triangulations and pull them back to the surface. Since the conformal mapping is angle preserving, and transforms infinitesimal circles on the surface to

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the planar infinitesimal circles, therefore it maps the planar Delaunay triangulations to the geodesic Delaunay triangulations on the surface. Furthermore, we can adapt the planar sampling density, such that the sampling on the surface normal cycle is sufficiently dense. The geodesic Delaunay property and the sampling density ensure the surface meshing quality.

The conformal flattending can be carried out using discrete surface Ricci flow theory, which is capalbe of handling surfaces with complicated topologies and guarantees the existence and the uniqueness of the solution. The planar meshing is based on Delaunay refinement and adaptive sampling, which guarantees the Delaunay property, the minimal angle and the sampling density.

2. THEORETIC FOUNDATION

This section briefly introduces the theoretic background for surface conformal geometry and surface Ricci flow. For more details, we refer readers to [8, 9, 6, 7] for more thorough treatments.

2.1 Discrete Surface Ricci Flow

Ricci flow deforms the Riemannian metric proportional to the Ricci curvature, such that the curvature evolves according to non-linear heat diffusion process and eventually converge to constants. Perelman used Ricci flow to prove Poincaré's conjecture [10, 11]. Hamilton [12] and Chow [13] proved the convergence of surface Ricci flow.

Smooth surface Ricci flow theory can be generalized to the discrete situation. A smooth surfaces is represented by a polyhedral surface S with vertex set V. We call (S, V) a *marked surface*. Given a marked surface, we can define different triangulations. A *discrete Riemannian metric* for a marked surface (S, V) with a triangulation T can be represented as edge lengths $d : E \to \mathbb{R}^+$, satisfying the triangle inequality, namely on each face $[v_i, v_j, v_k]$, $d(v_i, v_j) + d(v_j, v_k) > d(v_i, v_k)$. The discrete Riemannian metric determines the corner angles, by the cosine law

$$\cos\theta_i^{jk} = \frac{\left(d^2(v_i, v_j) + d^2(v_k, v_i) - d^2(v_j, v_k)\right)}{2d(v_i, v_j)d(v_k, v_i)}. \quad (1)$$

Fix a discrete Riemannian metric, there are many triangulations, among them, the *Delaunay triangulation* is highly preferred.

Definition 2.1 (Delaunay Triangulation)**.** Given a closed marked surface (S, V) with a discrete Riemannian metric d, a triangulation T is called Delaunay, if for any edge $[v_i, v_j]$ shared by two faces $[v_i, v_j, v_k]$ and $[v_j, v_i, v_l]$, the condition $\theta_k^{ij} + \theta_l^{ji} \leq \pi$ always holds.

A non-Delaunay triangulation can be modified to be Delaunay by a sequence of edge flip operators,

Definition 2.2 (Edge Flip Operator)**.** Given a closed marked surface (S, V) with a discrete Riemannian metric d and a triangulation T, an edge $[v_i, v_j]$ shared by two faces $[v_i, v_j, v_k]$ and $[v_j, v_i, v_l]$. The edge flip operator swaps the edge $[v_i, v_j]$ to $[v_k, v_l]$, and changes the triangles to $[v_i, v_l, v_k]$ and $[v_j, v_k, v_l]$.

It is well known that we can modify a triangulation to be Delaunay by a sequence edge flip operators, such that the longer diagonals are replaced by the shorter diagonals.

The *discrete Gaussian curvature* is defined as the angle deficit,

$$K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases} \quad (2)$$

The total discrete Gaussian curvature also satisfies the Gauss-Bonnet theorem,

$$\sum_{v_i \in \partial M} K(v_i) + \sum_{v_i \notin \partial M} K(v_i) = 2\pi\chi(M).$$

The conformal deformation is defined analogously

Definition 2.3 (Vertex Scaling)**.** Suppose $M = (V, E, F)$ is a triangulated polyhedral surface, with a discrete metric $l : E \to \mathbb{R}^+$. $\lambda : V \to \mathbb{R}$ is the discrete conformal factor function defined on the vertex set V, the vertex scaling operator is defined as follows:

$$l_{ij} \mapsto e^{u_i} l_{ij} e^{u_j}, \quad \forall\, [v_i, v_j] \in E. \quad (3)$$

The discrete conformal equivalence can be defined as follows: suppose d and d' are two polyhedral metrics of the marked surface (S, V), if there exists a sequence of triangulations T_i's and discrete metrics d_i's,

$$T_1, T_2, \ldots, T_n, \quad d = d_1, d_2, \ldots, d_n = d',$$

such that

1. T_i is Delaunay with respect to d_i;
2. if $T_i \neq T_{i+1}$ then they differ by an edge flip and $d_i = d_{i+1}$;
3. if $d_i \neq d_{i+1}$ then they differ by a vertex scaling and $T_i = T_{i+1}$.

Definition 2.4 (Discrete Conformal)**.** Two triangulated polyhedral metrics d and d' on a closed marked surface (S, V) are discrete conformal, if they are related by a sequence of two types of moves: vertex scaling and edge flip preserving Delaunay property.

The discrete surface Ricci flow is defined similar to the smooth one.

Definition 2.5 (Discrete Surface Ricci Flow)**.** Given a marked surface (S,V) with a polyhedral metric d and a triangulation T, suppose the target Gaussian curvature $\bar{K}: T \to \mathbb{R}$ is given, then the Ricci flow is defined as

$$\frac{d\lambda(v_i,t)}{dt} = \bar{K}(v_i) - K(v_i,t),$$

during the flow, the triangulation is updated to preserve the Delaunay property.

The discrete surface Ricci flow is gradient flow of the following convex Ricci energy:

$$E(\lambda) := \int^{\lambda} \sum_{i=1}^{n} (\bar{K}(v_i) - K(v_i)) d\lambda_i. \qquad (4)$$

The Hessian matrix of the Ricci energy can be represented by the cotangent edge weight, for all edge $[v_i, v_j]$. $w_{ij} = \cot\theta_k^{ij} + \cot_l^{ji}$, and the Hessain matrix is $H = (h_{ij})$

$$h_{ij} = \begin{cases} \sum_{k \neq i} w_{ik} & i = j \\ -w_{ij} & i \neq j \end{cases} \qquad (5)$$

The existence and the uniqueness of the solution to the discrete surface Ricci flow is proved in the following theorem.

Theorem 2.1 (Discrete Surface Flow [6])**.** *Given a polyhedral metric d on a closed marked surface (S,V), and target curvature $\bar{K}: V \to (-\infty, 2\pi)$, such that $\bar{K}$ satisfies the Gauss-Bonnet conditon $\sum K(v) = 2\pi\chi(S)$, there is a $\bar{d}$ discrete conformal to d, and $\bar{d}$ realized the curvature $\bar{k}$. $\bar{d}$ is unique update to a scaling, and can be obtained by the discrete surface Ricci flow.*

3. COMPUTATIONAL ALGORITHMS

3.1 Discrete Uniformization

First, we compute the discrete conformal metric by setting the target curvature satisfying the Gauss-Bonnet condition. The target curvatures for interior vertices are zeros, and those for the boundary vertices are constant.

Once the target metric is obtained, we can isometrically embed the whole mesh on the plane face by face.

3.2 Planar Delaunay Refinement

The input object is a two-dimensional polygonal domain Ω, possibly with holes and constraining edges and vertices inside the domain. The *boundary* $\partial\Omega$ is a set of vertices and edges which separates the interior of Ω from its exterior. $\partial\Omega$ is a planar straight line graph (PSLG). We want to generate a mesh $\mathcal{T}$ of Ω, such that $\mathcal{T}$ contains good quality triangles. In order to obtain a good quality mesh, it is necessary for $\mathcal{T}$ to include additional points, called *Steiner points*, vertices of the mesh that are not vertices of the input PSLG. We want the total number of Steiner points to be as small as possible.

Algorithm 1 Discrete Surface Ricci Flow

Require: Triangle mesh M, target curvature $\bar{K}$
Ensure: Discrete Conformal Factor λ

```
Initialize λ_i ← 0, for all v_i ∈ V
while true do
    Update edge length using vertex scaling Eqn. (3)
    Update triangulation to be Delauny by edge flips
    Update corner angles using Eqn. (1)
    Update vertex curvature using Eqn. (2)
    if max|K̄_i − K_i| < ε then
        Return λ
    end if
    Compute the gradient ∇E = (K̄_i − K_i)
    Compute the Hessian matrix H Eqn. (5)
    Solve linear system Hμ = ∇E
    Update the conformal factor λ ← λ − μ
end while
```

Various approaches have been developed for this purpose, such as advancing-front methods, quadtree methods, Delaunay-based methods, [14, 15], and the combinations of them. Most of them work well in practice but come with no guarantee on the quality and size of the generated mesh. The algorithm we use is Delaunay refinement proposed by Chew [14] and Ruppert [15]. It is a simple technique to incrementally placing Steiner points at the circumcenters of bad-quality Delaunay triangles.

A circumcenter of a triangle may lie outside the domain. When it happens, at least a boundary edge (segment) is very close to some existing vertices. Call a vertex *encroaches upon* a segment if it lies inside its diametrical circumcircle.

The algorithm is given in the Algorithm 2.

4. EXPERIMENTAL RESULTS

The proposed algorithm has been thoroughly tested on real CAD models, the initial meshes are extracted from NURBS. The remeshing results guarantees the minimal angle is greater than 26 degree and preserve all the sharp features as shown in Fig. 1 and Fig. 2.

Algorithm 2 Delaunay Refinement $(\Omega, \theta_{\min})$

Require: A 2d polygonal domain Ω; the desired minimal angle of output triangles $\theta_{\min}$
Ensure: A mesh $\mathcal{T}$ of Ω

```
Construct an initial Delaunay mesh T of ∂Ω;
while ∃τ ∈ T and MinAngle(τ) > θ_min do
    let c be the circumcenter of τ;
    if c encroaches upon any segment of T then
        split an encroached segment;
    else insert c into the Delaunay mesh T;
    end if
end while
```

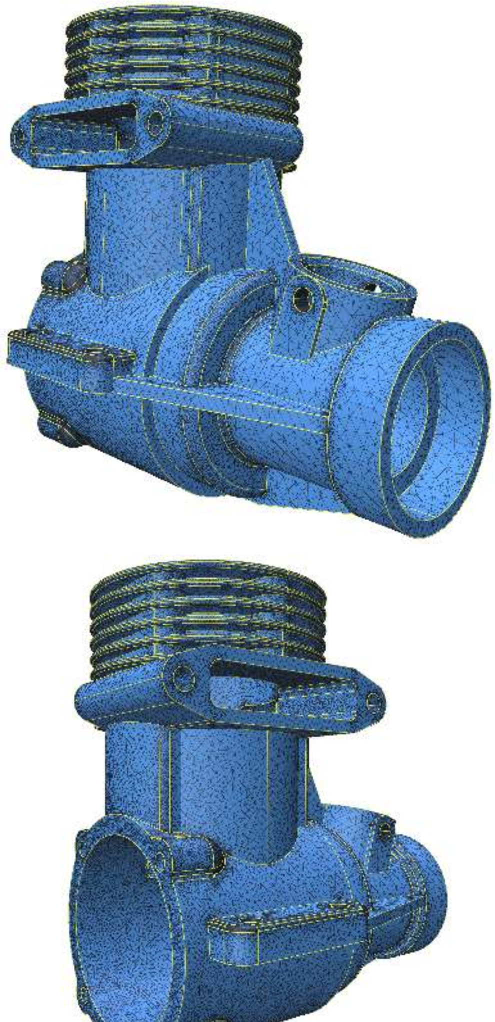

Figure 1: Remesh result for the Crank model.

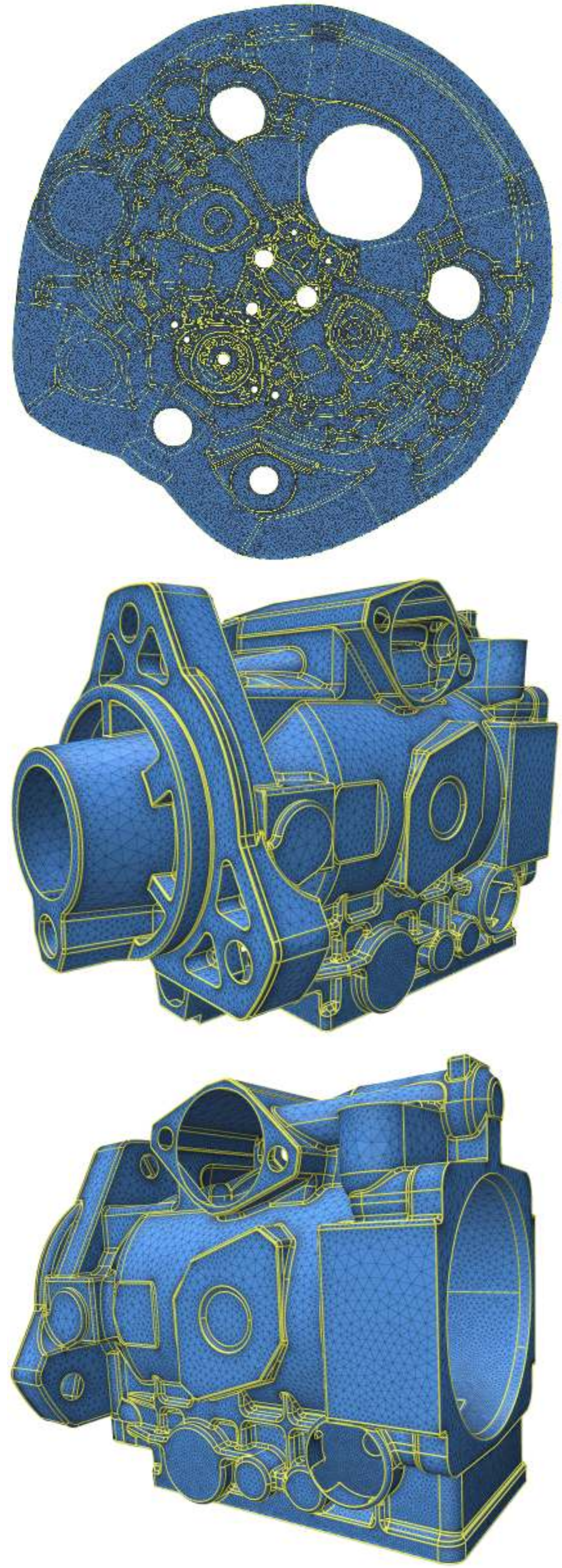

Figure 2: Conformal parameterization and remesh result.

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